

## Inverse tasks of electromagnetic field utilizing sensitivity analysis in the time domain

**Abstract.** In last years, the methods for material structure recognition using the modern CAD technology have made remarkable progress in achieving automatically optimum designs. The recognition and optimization of structure, are done in order to fine tune the optimum layout. For this aim, the measurements of desired field quantity for sufficient number of points have to be undertaken. Next, the inverse task, consisting in fact of repeated forward tasks, should be conducted.

**Streszczenie.** W ostatnich latach metody rozpoznawania struktury materiału z wykorzystaniem nowoczesnych metod komputerowych poczyniły znaczny postęp. Celem rozpoznawania struktury materiału przewodzącego jest optymalizacja jego kształtu lub ocena stanu tej struktury pod kątem obecności pęknięć i defektów. Punktem wyjścia jest dokonanie odpowiedniej liczby pomiarów pola magnetycznego nad powierzchnią przewodzącą, aby następnie przeprowadzenie zadania odwrotnego wyznaczenia struktury materiału przewodzącego. (Zadania odwrotne pola elektromagnetycznego wykorzystujące analizę wrażliwości w dziedzinie czasu)

**Keywords:** electromagnetism, finite element analysis, sensitivity analysis, numerical analysis.

**Słowa kluczowe:** elektromagnetyzm, metoda elementów skończonych, analiza wrażliwości, analiza numeryczna.

### Introduction

Deciding to use eddy current method to recognize the structure of conductive material, it is necessary to select the electric current shape in the coil used for field excitation. Harmonic excitation is simple but in order to take into account the response of eddy currents flowing at different depths need to use multiple frequencies. Pulse excitation causes measurement difficulties, but it provides a large amount of data necessary to identify the internal structure. Pulse excitation requires an analysis of the electromagnetic field in the time domain, which now allow all the commercial programs for this purpose. This work describes the inverse algorithm performed in the time domain, allowing reproduction of the material structure from measurements of the magnetic field outside.

### Problem definition

Magnetic field, exciting eddy-currents in conductive area, is caused by the coil powered with impulse current. The measurement coil, placed nearby exciting coil, may be translated to different positions, registering the shape of induced voltage. Time-discretization of this voltage, done for different positions provides large number of measurement data, which may be utilized by the optimizing algorithm. Using non-harmonic excitation currents, we dispose with redundant number of measurement data. From this reason, the algorithms for solution of over-determined equations systems have to be adopted. Because of measurement errors, the task is often ill-posed. In this case we propose application of regularization with singular value decomposition (SVD) of equations matrix. As result we obtain conductivity distribution in finite elements of conductive area. While the conductivity value inside finite elements remains constant, exactness of the recognition depends strongly from mesh refinement.

### Inverse task

Inverse task bases on sequential forward-solutions, while optimised parameters change according to chosen optimisation method. We use the Gauss-Newton algorithm, i.e. we minimize the form:

$$(1) \quad \min_{\Delta\gamma} \|\Delta V - S \cdot \Delta\gamma\|_2$$

where  $\Delta V$  means a difference between desired (measured) and simulated voltages,  $\Delta\gamma$  – conductivity corrections and

$S$  – sensitivity matrix. Applying an excess of measurement data causes the matrix  $S$  being non-quadratic, and in this case  $S$  will be decomposed using singular value decomposition. The main problem of inverse task is evaluation of sensitivity matrix, for described case as sensitivity of voltages on measurement coils calculated versus electric conductivity in finite elements. Although performing of inverse tasks utilizing gradient-free methods would also be possible, but such methods are much less efficient and obtaining a solution within a reasonable time of calculation is problematic.

Assuming that we dispose of matrix  $S$ , we can iteratively evaluate conductivity corrections, hoping the algorithm converges to measurement values. Reasons for the lack of convergence are usually the poorly matched measurement data. So, in order to improve the convergence we usually provide data in excess. Application of singular value decomposition afterwards allows to eliminate this data, which sensitivity versus conductivities is negligible.

### Singular value decomposition

SVD consists in decomposition of matrix  $S$  into three terms:

$$(2) \quad S = U \cdot D \cdot W^T \Rightarrow U \cdot D \cdot W^T \cdot \Delta\gamma = \Delta V,$$

with orthogonal matrices  $U$  and  $W$ , and diagonal matrix  $D$ . First we find vector  $X$

$$(3) \quad D \cdot X = U^T \cdot \Delta V.$$

Next,  $X$  may be truncated according to corresponding singular values of  $S$ . Truncation threshold is implied by user, causing decreasing of the rank of matrix  $S$ . Then, the corrections vector  $\Delta\gamma$  is found:

$$(4) \quad \Delta\gamma = W \cdot X.$$

Proposed approach allows to stabilize inverse algorithm. During the progressive iterations the rank of the matrix is increasing, allowing for exact recognition of conductivity distribution.

### Evaluation of sensitivity using FETD

Let's assume the following definition for sensitivity of magnetic vector potential in node  $i$  versus conductivity in element  $e$ :

$$(5) \quad S_{i,(e)} = \frac{\partial A_i}{\partial \gamma^{(e)}}$$

FE-solution basing on time-stepping method for magnetic vector potential  $A$  and excitation currents  $i$  relies on equations system:

$$(6) \quad \begin{aligned} \left[ \Theta \cdot \mathbf{K} + \frac{I}{\Delta t} \mathbf{M} \right] A_i = \\ = \left( \frac{I}{\Delta t} \mathbf{M} - (1-\Theta) \mathbf{K} \right) A_{i-1} + (1-\Theta) \mathbf{i}_i + \Theta \mathbf{i}_{i-1} \end{aligned}$$

where  $\mathbf{K}$  and  $\mathbf{M}$  are the stiffness and mass matrices of finite elements containing the material parameters and geometric properties of the simulated sample,  $A_i$  is the vector of the desired node values and  $\mathbf{i}_i$  is the discretized excitation for time steps  $i \cdot \Delta t$ , with  $i = 1, \dots, n$ , and the parameter  $\Theta$  determines the time stepping scheme. For  $\Theta = 0$  we have Euler's forward scheme,  $\Theta = 0,5$  Crank-Nicholson scheme,  $\Theta = 2/3$  Galerkin scheme and for  $\Theta = 1$  Euler's backward scheme. Direct differentiation of equation (6) versus  $\gamma$  gives sensitivity equation:

$$(7) \quad \begin{aligned} \left[ \Theta \cdot \mathbf{K} + \frac{I}{\Delta t} \mathbf{C} \right] S_{i,(e)} = \\ = \left( \frac{I}{\Delta t} \mathbf{M} - (1-\Theta) \mathbf{K} \right) S_{i-1,(e)} + \frac{I}{\Delta t} \frac{\partial \mathbf{M}}{\partial \gamma^{(e)}} (A_{i-1} - A_i) \end{aligned}$$

The method is very effective, because the equations (6) and (7) differ only in excitations (right hand side).

Competitive method is this basing on Tellegen's theorem, brightly described in [1]. It leads to following sensitivity equation:

$$(8) \quad \int_{\Omega} \int_0^T \mathbf{J}^+(\tau) \cdot \Delta \mathbf{E}(t) d\Omega dt = \int_{\Omega} \int_0^T \Delta \gamma \cdot \mathbf{E}^+(\tau) \cdot \mathbf{E}(t) d\Omega dt$$

On the left-hand side we have increment of electric intensity vector,  $\Delta \mathbf{E}$ , which allows us to calculate coil voltages. On the right-hand side are sensitivity components from particular elements multiplied by  $\Delta \gamma$ . The excitation of an adjoint model  $\mathbf{J}^+$  is directed to these nodes, where the sensitivity should be calculated. From this reason both models, original and adjoint, differ only in excitation and can be solved using the same matrix decomposition.

The main difficulty using above described method are time-stepping points, which should match in both models. The original model is analyzed in forward time  $t$ , while the adjoint model in backward time  $\tau$ , as shown in Fig.1. Matching of points for both analyses is forcing constant time step, which may be in most cases not optimal.

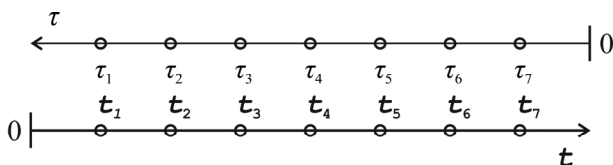


Fig.1. Forward and backward time of analysis

### The idea of semi-discrete method

Semi-discrete method allows to overcome problems of time-stepping. It bases on idea presented by Zienkiewicz in early 80ths. Drawback of this method are large and full matrices. As a method for analysis in time domain, it can not compete with finite elements utilizing time stepping. The

situation changes, when the sensitivity analysis is desired. The matrices are evaluated only once for both models: original and adjoint. The time points can be chosen arbitrary, because the solution is given as analytical time function.

### Semi-discrete finite element analysis

The diffusion equation we are solving has the form of

$$(9) \quad \mathbf{K} \cdot A(t) + \mathbf{M} \frac{\partial A(t)}{\partial t} = \mathbf{i}(t),$$

with:  $\mathbf{K}$  – stiffness and  $\mathbf{M}$  – mass matrix, while  $A$  means magnetic vector potential.  $A$  may be modified for solutions utilizing cylindrical symmetry of models. Excitation current  $\mathbf{i}(t)$  of adjoint model has the form of a unit step, suitable for sensitivity evaluation [1]. While solving (9) as the usual transient equation with zero initial conditions, we obtain the semi-discrete solution:

$$(10) \quad A(t) = \left( [I] - \exp(-t \cdot \mathbf{M}^{-1} \mathbf{K}) \right) \cdot \mathbf{K}^{-1} \cdot \mathbf{i}$$

When the inversion of mass matrix is necessary, we can not apply (10) for the whole region. It has to be subdivided into a conductive part "1" and a non-conductive part "2":

$$(11) \quad \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{Bmatrix} A_1(t) \\ A_2(t) \end{Bmatrix} + \begin{bmatrix} \mathbf{M}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial t} A_1(t) \\ \mathbf{0} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{i}_2 \end{Bmatrix}$$

We assume that excitation currents are located only in a non-conducting part of the region. Then we obtain the following matrix equation for the conducting region:

$$(12) \quad \begin{aligned} \left( \mathbf{K}_{11} - \mathbf{K}_{12} \cdot \mathbf{K}_{22}^{-1} \cdot \mathbf{K}_{21} \right) A_1(t) + \\ + \mathbf{M}_{11} \frac{\partial A_1(t)}{\partial t} = -\mathbf{K}_{12} \cdot \mathbf{K}_{22}^{-1} \cdot \mathbf{i}_2(t), \text{ or:} \\ \mathbf{K}_c \cdot A_1(t) + \mathbf{M}_{11} \frac{\partial A_1(t)}{\partial t} = \mathbf{i}_c(t). \end{aligned}$$

The similarity of (12) to (9) allows exploitation of solution (10). After differentiating, we obtain analytical formula (13) for the electric intensity vector  $\mathbf{E}$  in original and adjoint system (\*):

$$(13) \quad \begin{aligned} E_I(t) &= \frac{\partial A_I(t)}{\partial t} \\ &= \mathbf{M}_{11}^{-1} \exp(-t \cdot \mathbf{K}_c \cdot \mathbf{M}_{11}^{-1}) \cdot \mathbf{i}_c, \\ E_I^+(\tau) &= \frac{\partial A_I^+(\tau)}{\partial t} \\ &= \mathbf{M}_{11}^{-1} \exp(-\tau \cdot \mathbf{K}_c \cdot \mathbf{M}_{11}^{-1}) \cdot \mathbf{i}_c^+, \end{aligned}$$

with  $\tau + t = T$ .

### Semi-discrete sensitivity analysis

In further considerations, we are interested in calculating a sensitivity term on the right-hand side of (8):

$$(14) \quad \int_{\Omega} \int_0^T \mathbf{E}^+(\tau) \cdot \mathbf{E}(t) d\Omega dt$$

Integration of the product of electric intensity vectors over the finite element of first order requires the Gaussian quadrature of the second order. When Gauss points lie on

the element borders, combinations of nodal value products should be used. To obtain all necessary products, we define the following auxiliary matrix  $\mathbf{R}$ :

$$(15) \quad \mathbf{R} = \Delta \cdot \int_0^T \mathbf{E}^+(\tau) \cdot \mathbf{E}(t)^T dt = \\ = \int_0^T \mathbf{M}_{II}^{-1} \exp(-t \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) \cdot \mathbf{i}_c \cdot \\ \cdot \left( \mathbf{M}_{II}^{-1} \exp(-\tau \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) \cdot \mathbf{i}_c^+ \right)^T dt,$$

where  $\Delta$  is the size of finite element. The matrix  $\mathbf{R}$  can be rewritten as follows:

$$(16) \quad \mathbf{R} = \Delta \cdot \int_0^T \mathbf{M}_{II}^{-1} \exp(-t \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) \cdot \mathbf{i}_c \cdot \mathbf{i}_c^{+T} \cdot \\ \cdot \mathbf{M}_{II}^{-1} \exp(-\tau \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) dt$$

because the term  $\mathbf{M}_{II}^{-1} \exp(-\tau \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1})$  is symmetric.

A drawback of the established definition is that  $\mathbf{R}$  is singular (it contains a singular term  $\mathbf{I}_c = \mathbf{i}_c \cdot \mathbf{i}_c^{+T}$ ). For better numerical efficiency, analytical integration in the time would be essential. Since simple disentangling of exponential functions in (16) is impossible, we propose two following methods for this purpose.

### The Zassenhaus formula

The Zassenhaus formula [6] is a version of the Baker-Campbell-Hausdorff formula. It can be written involving commutator nesting:

$$(17) \quad \exp(\mathbf{X}) \cdot \mathbf{Y} \cdot \exp(-\mathbf{X}) = \\ = \mathbf{Y} + [\mathbf{X}, \mathbf{Y}] + \frac{1}{2!} [\mathbf{X}, [\mathbf{X}, \mathbf{Y}]] + \frac{1}{3!} [\mathbf{X}, [\mathbf{X}, [\mathbf{X}, \mathbf{Y}]]] + \dots,$$

$$\text{where: } [\mathbf{X}, \mathbf{Y}] = \mathbf{X} \cdot \mathbf{Y} - \mathbf{Y} \cdot \mathbf{X}.$$

In our case:

$$(18) \quad \mathbf{X} = -t \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1} \text{ and } \mathbf{Y} = \mathbf{i}_c \cdot \mathbf{i}_c^{+T} \cdot \mathbf{M}_{II}^{-1} = \mathbf{I}_c \cdot \mathbf{M}_{II}^{-1}.$$

If  $1/M$  is the norm of matrices  $\mathbf{X}$  and  $\mathbf{Y}$ , an error of the Zassenhaus approximation is of the order  $O(1/M^2)$ . It means that sensitivity calculations based on the Zassenhaus formula converge only for small times  $t$ , dependent on the norm of matrix  $\mathbf{K}_c \mathbf{M}_{II}^{-1}$ .

### Improvement of matrix commutations

Let us apply the Zassenhaus formula for two good commuting matrices. The result is then

$$(19) \quad \exp(\mathbf{X}) \cdot \mathbf{Y} \cdot \exp(-\mathbf{X}) = \mathbf{Y}, \text{ assuming: } [\mathbf{X}, \mathbf{Y}] = \mathbf{0}$$

The identity matrix commutes well with all other matrices. We add and subtract the identity matrix  $[\mathbf{I}]$  multiplied by an arbitrary large coefficient  $C$ :

$$(20) \quad \mathbf{R} = \Delta \cdot \int_0^T \mathbf{M}_{II}^{-1} \exp(-t \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) \cdot \\ \cdot \left( \mathbf{I}_c \cdot \mathbf{M}_{II}^{-1} + C \cdot [\mathbf{I}] - C \cdot [\mathbf{I}] \right) \cdot \exp(-\tau \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) dt.$$

Then, we divide our integral among two components:

$$\mathbf{R} = \Delta \cdot \int_0^T \mathbf{M}_{II}^{-1} \exp(-t \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) - \\ - \exp(\mathbf{L}\mathbf{A}) \cdot \exp(-\tau \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) dt \\ (21) \quad - \Delta \cdot \int_0^T \mathbf{M}_{II}^{-1} \exp(-t \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) \cdot C \cdot [\mathbf{I}] \cdot \\ \cdot \exp(t \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) \cdot \exp(-\tau \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) dt, \\ \text{where: } \mathbf{L}\mathbf{A} = \ln(\mathbf{I}_c \cdot \mathbf{M}_{II}^{-1} + C \cdot [\mathbf{I}]).$$

Now we can disentangle exponents in our formula and integrate it:

$$(22) \quad \mathbf{R} = T \cdot \mathbf{M}_{II}^{-1} \cdot \Delta \cdot \\ \cdot \left( \exp(-T \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1} + \mathbf{L}\mathbf{A}) - C \cdot \exp(-T \cdot \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1}) \right)$$

The formula (22) allows for simple and effective evaluation of terms necessary for sensitivity calculation basing on Tellegen method. The inversion of matrix  $\mathbf{M}_{II}$  may be omitted, if the terms of  $\mathbf{W}$  in exponent would be calculated from equations system:

$$(23) \quad \mathbf{K}_c \cdot \mathbf{M}_{II}^{-1} = \mathbf{W}, \\ \mathbf{M}_{II} \cdot \mathbf{W} = \mathbf{K}_c^T = \mathbf{K}_c.$$

The correctness of proposed method was confirmed by comparison with sensitivity values obtained with standard time-stepping method. For practical implementation the Zassenhaus formula converges for too short period of time, only at the beginning of the pulse, whereas the calculation basing on (22) reveals good agreement in the whole time range.

Field analysis performed during the iterative inverse task is, of course, also based on the semi-discrete method (11). Because the proposed method consists of matrix operations, the solution of (23) can be easily parallelized, as well as calculation of exponential functions in (22).

### Numerical example – crack recognition

This example shows the exploitation of sensitivity knowledge to identify the conductivity distribution inside the pipe wall. The eddy-current probe, consisting of three coils, moves inside the pipe with increments of 2.5 mm (Fig.2). In each position, the probe is excited and the voltage impulse in the measurement coil registered. For the purposes of simulation, this is equivalent to 17 locations for the probe with 100 time steps. The measurement was simulated with the help of FEM adding random 1% relative error. The search area consisted of 64 elements with a conductivity of  $\gamma = 2 \cdot 10^7$  S/m (Fig. 3a). This means that, in every iteration step, there were  $17 \cdot 64 \cdot 100 = 108\,800$  sensitivity values calculated providing gradient information for iterative Gauss-Newton with TSVD algorithm. The conductivity distribution, as well as crack shape, was correctly identified after 8 iterations.

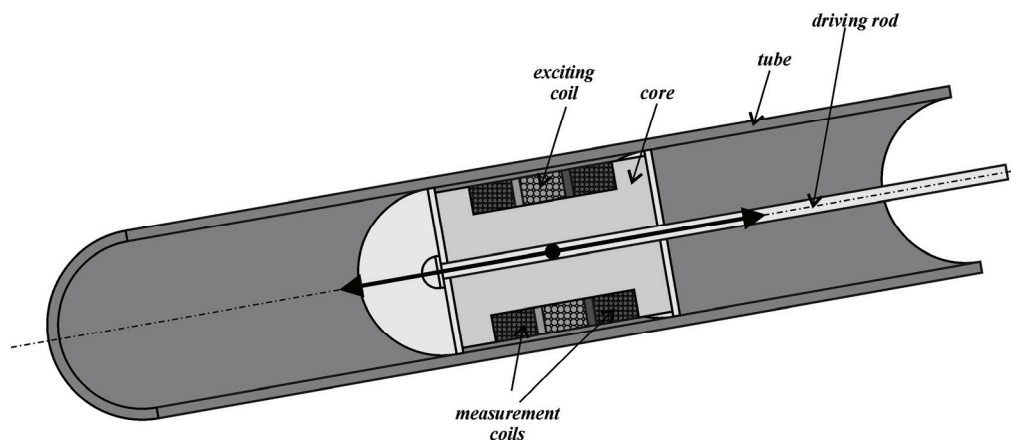


Fig.2. The sensor for tube testing consists of exciting and measurement coils

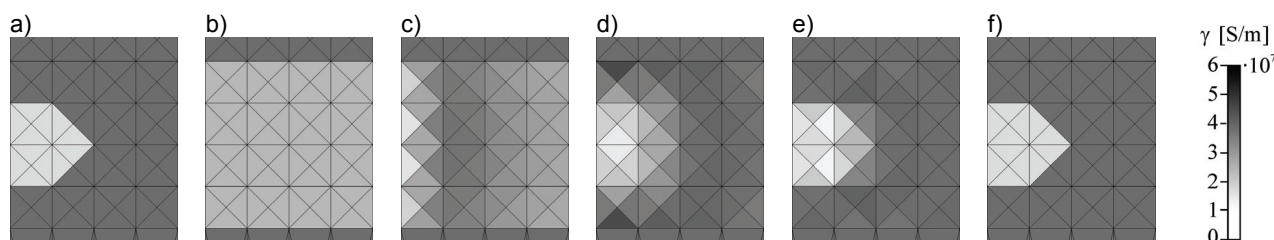


Fig.3. Identification process: a) assumed distribution of conductivity, b) initial distribution of conductivity, c) recognition of conductivity after 2 iteration, d) after 4 iteration, e) after 6 iteration, f) after 8 iteration

## Conclusions

The convergence of numerical identification algorithm described in the paper depends strongly on exact measurement of voltage's time function. In this work the measurement has been simulated using finite element model with additional noise. While using real measurement data the results of identification would be worse.

A very good choice is the solution of over-determined equation systems for the case of excess of measurement data. For identification of real cracks the application of data filtering and TSVD regularization of Gauss-Newton algorithm is necessary.

Comparison of the efficiency of the semi-discrete method with classical one shows that despite of high demand for memory, the described method may compete in relation to finite elements with the time stepping.

To analyze the wide class of real cracks the three-dimensional analysis should be applied. For 3D-formulation a new form of sensitivity equations has to be obtained. It will be aim of future work. The three dimensional algorithm will consume much more computation time, so it must be optimized carefully.

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