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Asymptotic stability of positive continuous-time linear systems with mutual state-feedbacks

Streszczenie. *W pracy sformułowano i rozwiązano problem stabilności dodatnich liniowych układów ciągłych ze wzajemnym sprzężeniem zwrotnym. Wykazano, że 1) jeżeli jeden z układów jest niestabilny wtedy układ ze sprzężeniem zwrotnym jest niestabilny dla wszystkich wartości tych sprzężeń, 2) jeżeli przynajmniej jeden element na głównej przekątnej blokowych macierzy jest dodatni to układ ze sprzężeniem zwrotnym jest niestabilny dla wszystkich wartości tych sprzężeń, 3) możliwość modyfikacji dynamiki przez dobór sprzężeń zwrotnych jest silnie ograniczona. Rozwiązanie ogólne zilustrowano dwoma przykładami. (**Stabilność asymptotyczna liniowych dodatnich układów ciągłych ze wzajemnymi sprzężeniami zwrotnymi**).*

Abstract. *A new problem of asymptotic stability of positive continuous-time linear systems coupled by mutual state-feedbacks is formulated. It is shown that: 1) If one of the coupled systems is unstable then the closed-loop system is unstable for all gain matrices of the mutual state-feedbacks, 2) If at least one diagonal entry of the block diagonal matrices is positive then the closed-loop system is unstable for all gain matrices, 3) the possibility of modification of the dynamics of the closed-loop system by suitable choice of gain matrices is strongly limited. The considerations are illustrated by two examples.*

Słowa kluczowe: stabilność asymptotyczna, ciągły, układ dodatni, sprzężenie zwrotne.
Keywords: asymptotic stability, continuous-time, linear system, positive, mutual state feedback.

Introduction

It is well-known [1, 2, 5, 7] that if the pair (*A*,*B*) of continuous-time linear system is controllable then by suitable choice of the gain matrix *K* of state-feedbacks it is possible to assign eigenvalues of the closed-loop system matrix *A+BK* in prescribed positions in the complex plane.

In this paper the asymptotic stability of positive continuous-time linear systems coupled by mutual state-feedbacks is addressed. It will be shown that if one of the coupled systems is unstable then the closed-loop system is unstable for all gain matrices of the mutual state-feedbacks. It will be also shown that the possibility of modification of the dynamics of the closed-loop system by suitable choice of gain matrices is strongly limited.

The paper is organized as follows. In section 2 some preliminaries on asymptotic stability of positive linear systems are recalled and formulation of the problem is given. The solution of the problem and the main result of the paper are presented in section 3. Concluding remarks are given in section 4.

Preliminaries and problem formulation

Let $\Re^{n\times m}$ be the set of $n\times m$ real matrices. The set $n\times m$ matrices with nonnegative entries will be denoted by $\Re_+^{n\times m}$ and $\Re_+^n = \Re_+^{n\times 1}$. The set of nonnegative integers will be denoted by Z_+ and the $n\times n$ identity matrix will be denoted by I_n.

Consider the linear continuous-time system

$$\dot{x} = Ax + Bu \tag{1}$$

where $x = x(t) \in \Re^n$ is the state vector, $u = u(t) \in \Re^m$ is input vector and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.

Definition 1. The system (1) is called positive if and only if $x(t) \in \Re_+^n$, $t \ge 0$ for any initial conditions $x(0) = x_0 \in \Re_+^n$ and all input vectors $u(t) \in \Re_+^m$, $t \ge 0$.

Theorem 1. [4] The system (1) is positive if and only if

$$A \in M_n \text{ and } B \in \Re_+^{n\times m} \tag{2}$$

where M_n is the set of $n\times n$ Metzler matrices, i.e. real matrices with nonnegative off diagonal entries.

Definition 2. The positive system (1) is called asymptotically stable if and only if

$$\lim_{t\to\infty} x(t) = 0 \text{ for all } x_0 \in \Re_+^n \tag{3}$$

Theorem 2. [3, 4 p.64, 6] The positive system (1) asymptotically stable if and only if its characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{4}$$

has all positive coefficients, i.e. $a_k > 0,\ k = 0,1,...,n-1$.

Theorem 3. [4 p.65] The positive system (1) is unstable if at least one diagonal entry of the matrix $A = [a_{ij}]$ is positive, i.e. $a_{ii} > 0$ for some $i \in (1,2,...,n)$.

Consider the positive linear continuous-time systems

$$\dot{x}_1 = A_{11}x_1 + B_1 u_1 \tag{5a}$$

and

$$\dot{x}_2 = A_{22}x_2 + B_2 u_2 \tag{5a}$$

where $x_1 \in \Re_+^{n_1}$ and $x_2 \in \Re_+^{n_2}$ are the state vectors, $u_1 \in \Re_+^{m_1}$ and $u_2 \in \Re_+^{m_2}$ are the input vectors and $A_{11} \in M_{n_1}$, $B_1 \in \Re_+^{n_1\times m_1}$, $A_{22} \in M_{n_2}$, $B_2 \in \Re_+^{n_2\times m_2}$.

Let

$$u_1 = K_2 x_2 \text{ and } u_2 = K_1 x_1 \tag{6}$$

where $K_1 \in \Re^{m_2\times n_1}$, $K_2 \in \Re^{m_1\times n_2}$ are gain matrices.

Substitution of (6) into (5) yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & B_1K_2 \\ B_2K_1 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{7}$$

The system (7) will be called the closed-loop system.
We are looking for K_1 and K_2 such that the closed-loop system (7) is positive and asymptotically stable.

Problem solution

Note that

(8) $$\begin{bmatrix} A_{11} & B_1K_2 \\ B_2K_1 & A_{22} \end{bmatrix} \in M_n \quad (n = n_1 + n_2)$$

if and only if the systems (5) are positive and

(9) $$K_1 \in \Re_+^{m_2 \times n_1}, \ K_2 \in \Re_+^{m_1 \times n_2}$$

Theorem 4. If the positive system (5a) or (5b) is unstable then the positive closed-loop system (7) is also unstable for all gain matrices (9).
Proof. The positive system (7) is asymptotically stable only if both matrices A_{11} and A_{22} are asymptotically stable. From the form of the matrix (8) it follows that its block diagonal matrices A_{11} and A_{22} are independent of the gain matrices K_1 and K_2. Therefore if the matrix A_{11} or A_{22} is unstable then the positive system (7) is unstable for all gain matrices (9).
□
Theorem 5. If at least one diagonal entry of the matrix A_{11} or of the matrix A_{22} is positive then the positive system (7) is unstable for all gain matrices (9).
Proof. If at least one diagonal entry of the matrix A_{11} or of the matrix A_{22} is positive then by Theorem 3 the positive system (5a) or (5b) is unstable. In this case by Theorem 4 the positive system (7) is unstable for all gain matrices (9).
□
From Theorem 4 and 5 it follows that the possibility of modification of the dynamics of the system (7) by suitable choice of the gain matrices (9) is strongly limited. If the system (5a) or (5b) is unstable then by suitable choice of the gain matrices (9) we are not able to stabilize the system (7). On the following examples we shall show the limits of modification of the dynamic of the system (7) by suitable choice of he gain matrices (9).
Example 1. Consider the positive systems (5) witch the matrices

(10) $$\begin{gathered} A_1 = [a_{11}], \quad B_1 = [b_1], \\ A_2 = [a_{22}], \quad B_2 = [b_2] \\ (b_1 > 0, b_2 > 0) \end{gathered}$$

Assuming $K_1 = [k_1], \ K_2 = [k_2]$ we obtain the characteristic polynomial of the closed-loop system (7) of the form

(11) $$\det\begin{bmatrix} I_{n_1}s - A_{11} & -B_1K_2 \\ -B_2K_1 & I_{n_2}s - A_{22} \end{bmatrix} = \begin{vmatrix} s - a_{11} & -b_1k_2 \\ -b_2k_1 & s - a_{22} \end{vmatrix}$$
$$= s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - b_1b_2k_1k_2$$

From (11) it follows that the positive closed-loop system is unstable for any values of k_1 and k_2 ($k_1 \ge 0, k_2 \ge 0$) if at least one of the coefficient of the polynomial is nonnegative. It is easy to show that for any value of $k_1 \ge 0$ and $k_2 \ge 0$ the characteristic polynomial (11) has only the real zeros

(12) $$s_1 = \frac{a_{11} + a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4b_1b_2k_1k_2}}{2},$$
$$s_2 = \frac{a_{11} + a_{22} - \sqrt{(a_{11} - a_{22})^2 + 4b_1b_2k_1k_2}}{2}$$

From (12) it follows that if $a_{11} < 0$, $a_{22} < 0$ and

(13) $$k_1k_2 < \frac{a_{11}a_{22}}{b_1b_2}$$

then the closed-loop system is asymptotically stable.
By suitable choice of k_1 and k_2 satisfying (13) we have limited possibility of changing of the positions of the zeros (12) on the left half of the complex plane.
Fig. 1 shows for $a_{22} > a_{11}$ the changes of positions of the zeros (12) when the coefficient $k = k_1k_2$ varies from $k = 0$ ($s_1 = a_{11}, s_2 = a_{22}$) to $k = k_0 = \frac{a_{11}a_{22}}{b_1b_2}$ ($s_1 = a_{11} + a_{22}, s_2 = 0$) $k \to k_0$

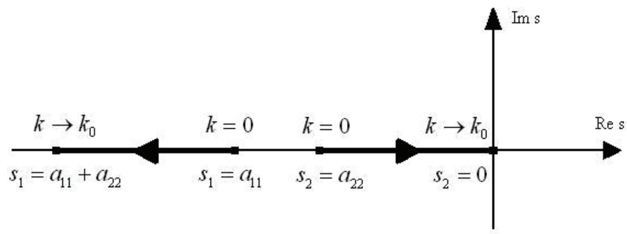

Fig. 1. The changes of positions of the zeros

Example 2. Consider two linear circuits shown on Fig. 2 with given resistances R_1, R_2, R_3, R, capacitances C_1, C_2, inductance L and source voltages e_1, e_2.

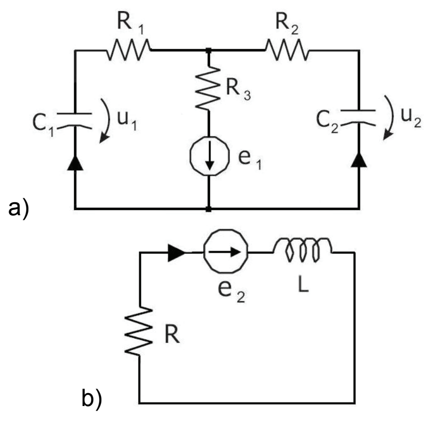

Fig. 2. Electrical circuits

Applying the Kirchhoff laws we may write the equations

(14a) $$e_1 = R_1C_1\frac{du_1}{dt} + u_1 + R_3(C_1\frac{du_1}{dt} + C_2\frac{du_2}{dt})$$

(14b) $$e_1 = R_2C_2\frac{du_2}{dt} + u_2 + R_3(C_1\frac{du_1}{dt} + C_2\frac{du_2}{dt})$$

for the first circuit (Fig. 2a) and the equation

(15) $$e_2 = Ri + L\frac{di}{dt}$$

for the second one (Fig. 2b).
The equations (14) can be written in the form

(16) $$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A_1\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B_1 e_1$$

where

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

(17) $$= \begin{bmatrix} -\frac{R_2+R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} & \frac{R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ \frac{R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} & -\frac{R_1+R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{R_2}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ \frac{R_1}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix}$$

From (15) we have

(18) $$\frac{di}{dt} = A_2 i + B_2 e_2$$

where

(19) $$A_2 = [a] = \left[-\frac{R}{L}\right],\ B_2 = [b] = \left[\frac{1}{L}\right]$$

The both electrical circuits are positive systems since A_1 and A_2 are Metzler matrices and the matrices B_1, B_2 have positive entries.
The electrical circuits are coupled by the mutual state-feedbacks

(20) $$e_1 = ki \text{ and } e_2 = [k_1 \quad k_2]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

In this case the matrix (8) has the form

(21) $$\begin{bmatrix} A_{11} & B_1K_2 \\ B_2K_1 & A_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_1k \\ a_{21} & a_{22} & b_2k \\ bk_1 & bk_2 & a \end{bmatrix}$$

and its characteristic polynomial

(22) $$p(s) = \det\begin{bmatrix} s-a_{11} & -a_{12} & -b_1k \\ -a_{21} & s-a_{22} & -b_2k \\ -bk_1 & -bk_2 & s-a \end{bmatrix} = s^3 + a_2s^2 + a_1s + a_0$$

where

(23) $$\begin{aligned} a_2 &= -(a_{11}+a_{22}+a), \\ a_1 &= a_{11}a_{22} + a_{11}a + a_{22}a - a_{12}a_{21} - bb_1kk_1 - bb_2kk_2, \\ a_0 &= a_{11}bb_2kk_2 + a_{22}bb_1kk_1 + a_{12}a_{21}a - a_{11}a_{22}a \\ &\quad - a_{21}bb_1kk_2 - a_{12}bb_2kk_1 \end{aligned}$$

If the gains k_1, k_2 and k are nonnegative then the matrix (21) is a Metzler matrix and its characteristic polynomial (22) has at least one real zero. It is easy to show that if at least one of the diagonal entries a_{11}, a_{22} and a is positive then at least one of the coefficients a_0, a_1, a_2 of the characteristic polynomial (22) is not positive.
Note that only the coefficients a_0 and a_1 depends on the gains k_1, k_2, k and the coefficient a_2 is independent of the gains. Therefore, by changing of the gains k_1, k_2, k we have limited influence on the locations of zeros of the characteristic polynomial (22).
For $R = R_1 = R_2 = R_3 = 1$, $C_1 = C_2 = 1$ and $L = 1$ the characteristic polynomial (22) has the form

(24) $$p(s) = \det\begin{bmatrix} s-a_{11} & -a_{12} & -b_1k \\ -a_{21} & s-a_{22} & -b_2k \\ -bk_1 & -bk_2 & s-a \end{bmatrix} = (s+1)\left(s^2 + \frac{4}{3}s + \frac{1}{3} - \frac{k}{3}(k_1+k_2)\right)$$

and its zeros are

(25) $$\begin{aligned} s_1 &= -1, \\ s_2 &= -\frac{2}{3} + \frac{1}{3}\sqrt{1+3k(k_1+k_2)}, \\ s_3 &= -\frac{2}{3}1\frac{1}{3}\sqrt{1+3k(k_1+k_2)} \end{aligned}$$

From (25) it follows that for nonnegative k_1, k_2, k only s_2 may be nonnegative for $k_1 + k_2 \geq \frac{1}{k}$.
Fig. 3 shows the asymptotic stability and instability regions on the plane k_1, k_2 for k satisfying the condition $0 < k_1 + k_2 < \frac{1}{k}$.

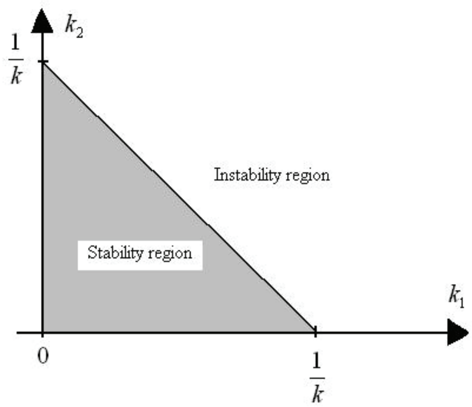

Fig. 3. The asymptotic stability and instability regions

The same result follows from the polynomial (22) since for asymptotic stable system the coefficient $\frac{1}{3}-\frac{k}{3}(k_1+k_2)$ should be positive.

Concluding Remarks

A new problem of asymptotic stability of positive continuous-time linear systems coupled by mutual state-feedbacks has been addressed. The following has been shown: 1) If one of the coupled systems is unstable then the closed-loop system (7) is unstable for all gain matrices K_1 and K_2 (Theorem 4). 2) If at least one diagonal entry of the matrices A_{11} or A_{22} is positive then the closed-loop system (7) is unstable for all gain matrices K_1 and K_2 (Theorem 5). 3) The possibility of modification of the dynamics of the closed-loop system (7) by suitable choice of gain matrices K_1 and K_2 is strongly limited. The considerations have been illustrated by two examples. In Example 1 Fig. 1 shows the changes of position of the zeros of the characteristic polynomial (11) when the coefficient k varies from $k=0$ to $k=k_0$. In Example 2 the asymptotic stability of two linear circuits coupled by state-feedbacks has been analyzed. These considerations can be easily extended for linear discrete-time systems coupled by mutual state-feedbacks.

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