Asymptotic stability of positive continuous-time linear systems with mutual state-feedbacks

**Abstract** A new problem of asymptotic stability of positive continuous-time linear systems coupled by mutual state-feedbacks is formulated. It is shown that: 1) If one of the coupled systems is unstable then the closed-loop system is unstable for all gain matrices of the mutual state-feedbacks, 2) If at least one diagonal entry of the block diagonal matrices is positive then the closed-loop system is unstable for all gain matrices, 3) the possibility of modification of the dynamics of the closed-loop system by suitable choice of gain matrices is strongly limited. The considerations are illustrated by two examples.

**Keywords:** asymptotic stability, continuous-time, linear system, positive, mutual state feedback.
The system (7) will be called the closed-loop system. We are looking for \( K_1 \) and \( K_2 \) such that the closed-loop system (7) is positive and asymptotically stable.

**Problem solution**

Note that

\[
\begin{bmatrix} A_1 & B_1 K_2 \\ B_2 K_1 & A_{22} \end{bmatrix} \in M_n \quad (n = n_1 + n_2)
\]

if and only if the systems (5) are positive and

\[
K_1 \in \mathbb{R}^{n_1 \times n_1}, \quad K_2 \in \mathbb{R}^{n_2 \times n_2}
\]

**Theorem 4.** If the positive system (5a) or (5b) is unstable then the positive closed-loop system (7) is also unstable for all gain matrices (9).

**Proof.** The positive system (7) is asymptotically stable only if both matrices \( A_{11} \) and \( A_{22} \) are asymptotically stable. From the form of the matrix (8) it follows that its block diagonal matrices \( A_{11} \) and \( A_{22} \) are independent of the gain matrices \( K_1 \) and \( K_2 \). Therefore if the matrix \( A_{11} \) or \( A_{22} \) is unstable then the positive system (7) is unstable for all gain matrices (9).

**Theorem 5.** If at least one diagonal entry of the matrix \( A_{11} \) or of the matrix \( A_{22} \) is positive then the positive system (7) is unstable for all gain matrices (9).

**Proof.** If at least one diagonal entry of the matrix \( A_{11} \) or of the matrix \( A_{22} \) is positive then by Theorem 3 the positive system (5a) or (5b) is unstable. In this case by Theorem 4 the positive system (7) is unstable for all gain matrices (9).

From Theorem 4 and 5 it follows that the possibility of modification of the dynamic of the system (7) by suitable choice of the gain matrices (9) is strongly limited. If the system (5a) or (5b) is unstable then by suitable choice of the gain matrices (9) we are not able to stabilize the system (7). On the following examples we shall show the limits of modification of the dynamic of the system (7) by suitable choice of he gain matrices (9).

**Example 1.** Consider the positive systems (5) witch the matrices

\[
A_1 = [a_{11}], \quad B_1 = [b_1], \\
A_2 = [a_{22}], \quad B_2 = [b_2] \\
(b_1 > 0, b_2 > 0)
\]

Assuming \( K_1 = [k_1] \), \( K_2 = [k_2] \) we obtain the characteristic polynomial of the closed-loop system (7) of the form

\[
\det \begin{bmatrix} I_{n_1} s - A_{11} & -B_1 K_2 \\ -B_2 K_1 & I_{n_2} s - A_{22} \end{bmatrix} = s^2 - (a_{11} + a_{22}) s + a_{11} a_{22} - b_1 b_2 k_1 k_2
\]

From (11) it follows that the positive closed-loop system is unstable for any values of \( k_1 \) and \( k_2 \) \((k_1 \geq 0, k_2 \geq 0)\) if at least one of the coefficient of the polynomial is nonnegative. It is easy to show that for any value of \( k_1 \geq 0 \) and \( k_2 \geq 0 \) the characteristic polynomial (11) has only the real zeros

\[
s_1 = \frac{a_{11} + a_{22} + \sqrt{(a_{11} - a_{22})^2 + 4b_1 b_2 k_1 k_2}}{2}, \\
s_2 = \frac{a_{11} + a_{22} - \sqrt{(a_{11} - a_{22})^2 + 4b_1 b_2 k_1 k_2}}{2}
\]

From (12) it follows that if \( a_{11} < 0, \ a_{22} < 0 \) and

\[
k_1 k_2 < \frac{a_{11} a_{22}}{b_1 b_2}
\]

then the closed-loop system is asymptotically stable.

By suitable choice of \( k_1 \) and \( k_2 \) satisfying (13) we have limited possibility of changing of the positions of the zeros (12) on the left half of the complex plane.

**Example 2.** Consider two linear circuits shown on Fig. 2

![Fig. 2. Electrical circuits](image)

Applying the Kirchhoff laws we may write the equations

\[
\begin{align}
(14a) \quad \dot{e}_1 &= R_1 C_1 \frac{du_1}{dt} + u_1 + R_3 (C_1 \frac{du_1}{dt} + C_2 \frac{du_2}{dt}) \\
(14b) \quad \dot{e}_1 &= R_2 C_2 \frac{du_2}{dt} + u_2 + R_3 (C_1 \frac{du_1}{dt} + C_2 \frac{du_2}{dt})
\end{align}
\]

for the first circuit (Fig. 2a) and the equation

Fig. 1. The changes of positions of the zeros

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**Note:** The content includes mathematical expressions and diagrams, which are essential for understanding the context of the problem. The text is formatted to maintain readability and coherence, with proper alignment and spacing to facilitate comprehension.
where

\[ A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{R_2 + R_3 + R_3} \\ \frac{R_2}{C_1[R_1(R_2 + R_3 + R_3) + R_2R_3] - C_2[R_1(R_2 + R_3 + R_3) + R_2R_3]} \end{bmatrix}, \]

\[ B_2 = \frac{R_2}{C_1[R_1(R_2 + R_3 + R_3) + R_2R_3] - C_2[R_1(R_2 + R_3 + R_3) + R_2R_3]} \]

From (15) we have

\[ \frac{di}{dt} = A_2 i + B_2 e_2 \]

where

\[ A_2 = [a] = \begin{bmatrix} -\frac{R}{L} \\ 1 \end{bmatrix}, \quad B_2 = [b] = \begin{bmatrix} 1 \\ L \end{bmatrix} \]

The both electrical circuits are positive systems since \( A_1 \) and \( A_2 \) are Metzler matrices and the matrices \( B_1, B_2 \) have positive entries. The electrical circuits are coupled by the mutual state-feedbacks

\[ e_1 = k_1 i \quad \text{and} \quad e_2 = [k_1 \ k_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \]

In this case the matrix (8) has the form

\[ \begin{bmatrix} A_1 & B_1 K_2 \\ B_2 K_1 & A_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_1k \\ a_{21} & a_{22} & b_2 \end{bmatrix} \]

and its characteristic polynomial

\[ p(s) = \det \begin{bmatrix} s - a_{11} & -a_{12} & -b_1k \\ -a_{21} & s - a_{22} & -b_2k \\ -b_1k & -b_2k & s - a \end{bmatrix} = s^3 + a_2 s^2 + a_1 s + a_0 \]

where

\[ a_2 = -(a_{11} + a_{22} + a), \]

\[ a_1 = a_{11}a_{22} + a_{12}a_{21} - b_1kk_1 - b_2kk_2, \]

\[ a_0 = a_{11}bb_2kk_2 + a_{22}bb_1kk_1 + a_{12}a_{21}a - a_{11}a_{22}a \]

\[-a_{21}bb_1kk_1 - a_{12}bb_2kk_2 \]

If the gains \( k_1, k_2 \) and \( k \) are nonnegative then the matrix (21) is a Metzler matrix and its characteristic polynomial (22) has at least one real zero. It is easy to show that if at least one of the diagonal entries \( a_{11}, a_{22} \) and \( a \) is positive then at least one of the coefficients \( a_0, a_1, a_2 \) of the characteristic polynomial (22) is nonpositive. Note that only the coefficients \( a_0 \) and \( a_1 \) depends on the gains \( k_1, k_2, k \) and the coefficient \( a_2 \) is independent of the gains. Therefore, by changing of the gains \( k_1, k_2, k \) we have limited influence on the locations of zeros of the characteristic polynomial (22).

For \( R = R_1 = R_2 = R_3 = 1 \), \( C_1 = C_2 = 1 \) and \( L = 1 \) the characteristic polynomial (22) has the form

\[ p(s) = \text{det} \begin{bmatrix} s - a_{11} & -a_{12} & -b_1k \\ -a_{21} & s - a_{22} & -b_2k \\ -b_1k & -b_2k & s - a \end{bmatrix} = (s + 1) \left( s^2 + \frac{4}{3}s + 1 - \frac{k}{3}(k_1 + k_2) \right) \]

and its zeros are

\[ s_1 = -1, \]

\[ s_2 = -\frac{2}{3} + \frac{1}{3}\sqrt{1 + 3k(k_1 + k_2)}, \]

\[ s_3 = -\frac{2}{3} - \frac{1}{3}\sqrt{1 + 3k(k_1 + k_2)} \]

From (25) it follows that for nonnegative \( k_1, k_2, k \) only \( s_2 \) may be nonnegative for \( k_1 + k_2 \geq \frac{1}{k} \).

Fig. 3 shows the asymptotic stability and instability regions on the plane \( k_1, k_2 \) for \( k \) satisfying the condition

\[ 0 < k_1 + k_2 < \frac{1}{k} \].
The same result follows from the polynomial (22) since for asymptotic stable system the coefficient $\frac{1}{3} - \frac{k}{3}(k_1 + k_2)$ should be positive.

**Concluding Remarks**

A new problem of asymptotic stability of positive continuous-time linear systems coupled by mutual state-feedbacks has been addressed. The following has been shown: 1) If one of the coupled systems is unstable then the closed-loop system (7) is unstable for all gain matrices $K_1$ and $K_2$ (Theorem 4). 2) If at least one diagonal entry of the matrices $A_{11}$ or $A_{22}$ is positive then the closed-loop system (7) is unstable for all gain matrices $K_1$ and $K_2$ (Theorem 5). 3) The possibility of modification of the dynamics of the closed-loop system (7) by suitable choice of gain matrices $K_1$ and $K_2$ is strongly limited. The considerations have been illustrated by two examples. In Example 1 Fig. 1 shows the changes of position of the zeros of the characteristic polynomial (11) when the coefficient $k$ varies from $k = 0$ to $k = k_0$. In Example 2 the asymptotic stability of two linear circuits coupled by state-feedbacks has been analyzed. These considerations can be easily extended for linear discrete-time systems coupled by mutual state-feedbacks.

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**LITERATURA**


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