Analysis of borderline cases of electromagnetic field variational problem

Abstract. The paper offers analysis of expression of energetic functional for formulation of electromagnetic field variational problem in nonlinear anisotropic media. Additional conditions applied to the expression result in receiving borderline cases in the form of separate stationary problems. The results are compared with analogical tasks of mathematical physics.

Streszczenie. W artykule przedstawiono rozwinięcie funkcjonalu energetycznego dla wariacyjnego sformułowania zadania analizy pola elektromagnetycznego w nieliniowym środowisku anizotropowym. Dodatkowe ograniczenia narzucone na otrzymane rozwiązanie pozwalają uzyskać sformułowanie przypadków szczególnych w formie oddzielnych zadań stacjonarnych. Wyniki porównano z analogicznymi zadaniami fizyki matematycznej. (Rozwinięcie funkcjonalu energetycznego dla wariacyjnego sformułowania zadania analizy pola elektromagnetycznego w nieliniowym środowisku anizotropowym)

Keywords: variation methodology, energetic functional, action function, electromagnetic field.

Słowa kluczowe: metoda wariacyjna, energetyczny funkcjonał, działanie po Hamiltonu, pole elektromagnetyczne.

Introduction

There are phenomena of different nature that may take place in a physical system. They occur as a result of transformation of energies of various types (electromagnetic, mechanical, thermal etc) in dielectric, semiconducting, and conductive media. According to the principle of Hamilton-Ostrogradsky, the real motion of a system between two given points differs from all other motions in the fact that for the real motion the variation of action function is equal to zero. The measurement unit of the action function is the measurement unit of energy multiplied by the measurement unit of time because the action function is the time integral of the Lagrangian which is defined as the kinetic energy of the system minus its potential energy.

In our paper we will show that the minimization of other expression (namely the expression of full energy of the system where the sum of all kinds of energies is used) gives us the same results. It allows us to avoid the time integration of Lagrange function.

Initial assumptions

Let us write down the full energy expression for an anisotropic nonlinear unhomogenous hysteresis medium where we must remember that the differentials in the equation are not full differentials; they characterize only the time change of the vectors \( \mathbf{D} \) and \( \mathbf{B} \).

As a result of mathematical transformations the following energy expression has been obtained [2]:

\[
W = \int D \cdot \frac{\partial}{\partial x} B \, dx - \int \left( \frac{\partial}{\partial x} D \right) \cdot \mathbf{J} \cdot \frac{\partial}{\partial x} \mathbf{A},
\]

where \( C \) is a constant that depends only on initial and boundary conditions.

The variation of the functional (2) in the case of scalar electric potential variation can be obtained in the form

\[
\delta W = \int \left( D + \mathbf{J} \right) \cdot \delta \mathbf{A} \cdot dV + \int \rho_0 \delta \varphi \cdot dV,
\]

Let us apply the formulas of vector analysis \( \text{div} (\nabla X) = \chi \text{div} X + \chi \text{div} X \) and \( \int \text{div} X \, dV = \int \mathbf{X} \, dS \) and transform the expression (3) into

\[
\delta W = \int \left( D + \mathbf{J} \right) \cdot \delta \mathbf{A} \cdot dV + \int \rho_0 \delta \varphi \cdot dV = \int \left( D + \mathbf{J} \right) \cdot \delta \mathbf{A} \cdot dV - \int \chi \text{div} X \, dS.
\]

Such problem is called mixed (or initial-boundary value problem) because both boundary and initial values are given.

Variation of energetic functional

When we accept the formulated assumptions, the energy expression consists of the three parts: the energy of electric field, the energy of magnetic field, and the work done to displace charged particles (due to the parts with finite electrical conductivity and existence of free particles) [2]

\[
W = \int D \cdot \frac{\partial}{\partial x} B \, dx - \int \left( \frac{\partial}{\partial x} D \right) \cdot \mathbf{J} \cdot \frac{\partial}{\partial x} \mathbf{A}.
\]
In these transformations we have applied the condition that \( \delta \varphi \equiv 0 \) on the boundary \( S \) as well as the known formula
\[
div \bar{J} = -\varphi \big/ \varphi_{t}.
\]

The variation of the functional (2) in the case of vector magnetic potential variation can be obtained in the form
\[
(5) \quad \delta A_W = \int \bar{H} \cdot \delta \bar{B} \cdot dV - \int \frac{(\dot{\bar{E}}/\varphi_{t} + \bar{J})\delta \bar{A} \cdot dV}{S}
\]

Let us apply the formulas of vector analysis \( \text{div}(\bar{X} \times \bar{Y}) = S_{\text{rot}} X \times Y \) and \( \int \text{div} \bar{X} dV = \frac{1}{S} \bar{X} dS \) and transform the expression (5) into
\[
(6) \quad \delta A_W = \int \text{div}(\bar{E} \times \delta \bar{A}) dV + \int \varphi \text{rot} \bar{H} \delta \bar{A} \cdot dV - \int \frac{(\dot{\bar{E}}/\varphi_{t} + \bar{J})\delta \bar{A} \cdot dV}{S} - \frac{1}{V} \int (\varphi \text{rot} \bar{E} - \bar{E}/\varphi_{t} - \bar{J}) \delta \bar{A} \cdot dV
\]

In these transformations we have applied the condition that \( \delta \bar{A} = 0 \) on the boundary \( S \).

The minimum of the functional (2) can be found when we make its variation (4) and (6) equal to zero, i.e.
\[
\delta \varphi W = 0; \quad \delta A_W = 0;
\]
\[
\int \frac{(-\text{div} \bar{D} + \varphi)\delta \varphi \cdot dV}{V} = 0; \quad \int \frac{(\varphi \text{rot} \bar{E} - \bar{E}/\varphi_{t} - \bar{J})\delta \bar{A} \cdot dV}{V} = 0.
\]

Since the variations \( \delta \varphi \) and \( \delta \bar{A} \) are arbitrary, the minimum can be achieved only when integrands be equal to zero:
\[
(8) \quad \text{div} \bar{D} = \varphi ; \quad \text{rot} \bar{H} = \bar{E} / \varphi_{t}.
\]

It can be concluded that functions which supply a minimal value to the energetic functional (2) are functions that satisfy the first and third Maxwell’s equations.

Let us obtain the separate expressions of the functional for borderline cases.

**Formulation of electrostatics variational problem**

The conditions which determine the existence of an electrostatic field are following:
\[
(9) \quad \varphi = \varphi_{0} = \text{const} ; \quad \bar{J} = 0.
\]

It may be admitted that before the moment \( t = t_{0} \) no electromagnetic field existed in the medium of volume \( V \). Then at the moment \( t = t_{0} \) the electrical current of density \( \bar{J} \) was caused to emerge what called to existence of a stationary electric field (as well as a stationary electromagnetic field). This dictates the following expression of the functional (2)
\[
(15) \quad W_{\varphi s} = \int \frac{dV}{V} \left[ \bar{H} \bar{H} - \frac{1}{\varphi} \bar{E} \bar{E} \right] d\varphi_{0} + C = \int \frac{dV}{V} \left[ \bar{H} \bar{H} - \frac{1}{\varphi} \bar{E} \bar{E} \right] d\varphi_{0} + \text{const}
\]

Variation of the functional (15) can be written down in the form
\[
(16) \quad \delta W_{\varphi s} = \int \frac{dV}{V} \left[ \bar{H} \delta \bar{B} - \frac{1}{\varphi} \bar{E} \delta \bar{A} \right] d\varphi_{0} = \int \frac{dV}{V} \left[ \bar{H} \delta \bar{B} - \frac{1}{\varphi} \bar{E} \delta \bar{A} \right] d\varphi_{0} = \int \frac{dV}{V} \left[ \bar{H} \delta \bar{B} - \frac{1}{\varphi} \bar{E} \delta \bar{A} \right] d\varphi_{0}
\]

In the transformations (16) we have applied the condition that \( \delta \bar{A} = 0 \) on the boundary \( S \).

The minimum of the functional (15) can be found when we make its variation (16) equal to zero, i.e.
\[
(17) \quad \delta W_{\varphi s} = 0; \quad \int \left[ \frac{dV}{V} \left( \bar{H} \delta \bar{B} - \bar{E} \delta \bar{A} / \varphi_{t} \right) \right] d\varphi_{0}.
\]

Since the variation \( \delta \varphi \) is arbitrary, the minimum can be achieved only when the integrand be equal to zero:
\[
(13) \quad \text{div} \bar{D} = \varphi.
\]

It can be concluded that function which supplies a minimal value to the energetic functional (10) is a function that satisfies the first Maxwell’s equation.

**Formulation of Magnetostatics Variational Problem**

The conditions which determine the existence of a magnetostatic field are following:
\[
(14) \quad \varphi = \varphi_{0} ; \quad \bar{J} = \text{const}.
\]

It may be admitted that before the moment \( t = t_{0} \) no electromagnetic field (or only electrostatic field) existed in the medium of volume \( V \). Then at the moment \( t = t_{0} \) the electrical current of density \( \bar{J} \) was caused to emerge what called to existence of a magnetostatic field (as well as a stationary electric field). This dictates the following expression of the functional (2)
\[
(19) \quad W_{ms} = \int \frac{dV}{V} \left[ \bar{H} \delta \bar{B} - \frac{1}{\varphi} \bar{E} \delta \bar{A} \right] d\varphi_{t} + C = \int \frac{dV}{V} \left[ \bar{H} \delta \bar{B} - \frac{1}{\varphi} \bar{E} \delta \bar{A} \right] d\varphi_{t} + \text{const}
\]

Variation of the functional (19) can be written down in the form
\[
(20) \quad \delta W_{ms} = \int \frac{dV}{V} \left[ \bar{H} \delta \bar{B} - \frac{1}{\varphi} \bar{E} \delta \bar{A} \right] d\varphi_{t} = \int \frac{dV}{V} \left[ \bar{H} \delta \bar{B} - \frac{1}{\varphi} \bar{E} \delta \bar{A} \right] d\varphi_{t} = \int \frac{dV}{V} \left[ \bar{H} \delta \bar{B} - \frac{1}{\varphi} \bar{E} \delta \bar{A} \right] d\varphi_{t}
\]

In the transformations (16) we have applied the condition that \( \delta \bar{A} = 0 \) on the boundary \( S \).

The minimum of the functional (15) can be found when we make its variation (15) equal to zero, i.e.
\[
(18) \quad \text{rot} \bar{H} = \bar{J}.
\]

It can be concluded that function which supplies a minimal value to the energetic functional (15) is a function that satisfies the first Maxwell’s equation.

**Formulation of stationary field variational problem**

The conditions which determine the existence of a magnetostatic field are following:
\[
(14) \quad \varphi = \varphi_{0} ; \quad \bar{J} = \text{const}.
\]

It may be admitted that before the moment \( t = t_{0} \) no electromagnetic field (or only electrostatic field) existed in the medium of volume \( V \). Then at the moment \( t = t_{0} \) the electrical current of density \( \bar{J} \) was caused to emerge what called to existence of a magnetostatic field (as well as a stationary electric field). This dictates the following expression of the functional (2)
\[
(19) \quad W_{st} = \int \frac{dV}{V} \left[ \bar{H} \delta \bar{B} - \frac{1}{\varphi} \bar{E} \delta \bar{A} \right] d\varphi_{t} + C = \int \frac{dV}{V} \left[ \bar{H} \delta \bar{B} - \frac{1}{\varphi} \bar{E} \delta \bar{A} \right] d\varphi_{t} + \text{const}
\]

Variation of the functional (19) can be written down in the form
The expression of time derivative can be written using a finite element:

\[ \frac{d}{dt} \phi \cdot dV = \int (\bar{D} + \bar{J}(t-t_0)) \text{grad} \phi \cdot dV \]

where \( a_0 \) is the coefficient of the back differentiation formula that depends on its order and the step of time integration; \( \bar{C}_D = iC_{Dx} + jC_{Dy} + kC_{Dz} \) is an element that depends on the values of \( \bar{D} \) on previous integration steps as well as on the order of the back integration formula and the step of time integration. The expression of time integral can be written using a quadrature integration formula:

\[ \int_{t_0}^{t} \bar{J} dt = b_0 \bar{J} + \bar{C}_J, \]

where \( b_0 \) is the coefficient of the quadrature formula that depends on its order and the step of time integration; \( \bar{C}_J = iC_{Jx} + jC_{Jy} + kC_{Jz} \) in the element that depends on the values of \( \bar{J} \) on previous integration steps as well as on the order of the quadrature formula and the step of time integration.

We will conduct the minimization of the functional (24) by means of invariant approximations methodology [1].

The distribution of any variable (either scalar or vector) inside a finite element is approximated by an appropriate Taylor’s polynomial [1] of \( n \)-th order in the form

\[ X_{\pm}(x,y,z) = \bar{T}m_{\pm} \tilde{X}_{\pm} = \bar{K}(x,y,z) \tilde{X}_{\pm}; \]

\[ X = A, B, H, J, D; \quad \xi = x, y, z; \]

\[ \phi(x,y,z) = \bar{T}m_{\pm} \tilde{\phi}_{m} = \bar{K}(x,y,z) \tilde{\phi}_{m}; \]

\[ \text{grad} \phi(x,y,z) = \bar{T}N_{m} \tilde{\phi}_{m} = \bar{R}_{vm}(x,y,z) \tilde{\phi}_{m}, \]

\[ \text{div} \bar{D}(x,y,z) = \bar{T}N_{m} \text{div} \bar{D} = \bar{R}_{vm}(x,y,z) \cdot \bar{D}_{m}, \]

\[ \text{rot} \bar{H}(x,y,z) = \bar{T}N_{m} \times \bar{H}_{m} = \bar{R}_{vm}(x,y,z) \times \bar{H}_{m}, \]

\[ B_x = -\partial A_y / \partial z + \partial A_z / \partial y = -\bar{T}N_{x} T^{-1}_{m} \tilde{A}_{x,m} + \bar{T}N_{y} T^{-1}_{m} \tilde{A}_{y,m}; \]

\[ B_z = -\partial A_x / \partial y + \partial A_y / \partial x = -\bar{T}N_{y} T^{-1}_{m} \tilde{A}_{y,m} + \bar{T}N_{z} T^{-1}_{m} \tilde{A}_{z,m}; \]

where \( \bar{T} \) - Taylor’s vector of a point located inside the \( m \)-th finite element.; \( T_{m} \) - inverse Taylor’s matrix for \( m \)-th finite element; \( \tilde{X}_{\pm} \) - nodal columns of \( m \)-th finite element; \( \bar{N} = \bar{T}N_{x} + j\bar{T}N_{y} + k\bar{T}N_{z} \) - Hamilton’s matrix; \( \bar{R}_{vm} = \bar{K}_{vm} + j\bar{K}_{ym} + k\bar{K}_{zm} \) - difference analogy of Hamilton’s operator in the linear space where functions are uniquely determined by their nodal columns.

**Conclusion**

The expression of energetic functional for variational problem of electromagnetic field analysis has been proposed as the total of the energy of electric field, the energy of magnetic field and the work done to displace charged particles. The application of different specific conditions leads to borderline cases of the task of electromagnetic field analysis. It is shown how to converse this expression into its algebraic counterpart on the basis of invariant approximations methodology. The minimization of the functional gives us solutions of the first and the third Maxwell’s equations.

**REFERENCES**


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