Cylindrical conductor in an arbitrary time-harmonic transverse magnetic field

Abstract. A long cylindrical conductor in an arbitrary time-harmonic transverse magnetic field is considered in the paper. The source magnetic vector potential near the conductor is expressed as power series in the cylindrical coordinates. By using the method of separation of variables, the magnetic field inside and outside the conductor as well as the eddy currents and Joule’s power losses were determined. The dependency of Joule’s power losses due to n-th spatial harmonic was investigated as a function of various parameters.

Streszczenie. Rozważono długi cylindryczny przewodnik w dowolnym poprzecznym harmonicznym polu magnetycznym. Potencjał wektorowy źródłowego pola magnetycznego przedstawiono w postaci szeregu potęgowego i metodą rozdzielenia zmiennych określono pole magnetyczne, prądy wirowe oraz straty mocy. Przeanalizowano zależność strat mocy od n-tej harmonicznej przestrzennej w funkcji różnych parametrów. (Długi cylindryczny przewodnik w dowolnym poprzecznym harmonicznym polu magnetycznym)

Keywords: time-harmonic magnetic field, eddy currents, power losses, cylindrical conductor.

Słowa klucze: pola magnetyczne, prądy wirowe, straty mocy, przewodniki cylindryczne.

Introduction

Infinitely long, straight conductors of constant cross-section are often models of many real configurations with electromagnetic field, e.g. power lines, transmission lines, material samples in induction heating. Such configurations have been widely discussed in literature, e.g. [1-5]. This paper is a generalization of the considerations from [1-3], which concern an infinitely long, straight wire of circular cross-section exposed to a time harmonic magnetic field generated by other long conducting region, and exposed to a time harmonic magnetic field inside and outside the conductor as well as the eddy currents and Joule’s power losses were determined. The dependency of Joule’s power losses due to n-th spatial harmonic was investigated as a function of various parameters.

Problem description

A long conductor of circular cross-section of radius \( a \) is located along \( z \) axis of cylindrical coordinates in a non-conducting region, and exposed to a time harmonic transverse magnetic field generated by other long conductors parallel to the considered one (see Fig. 1).

![Fig. 1. Long cylindrical conductor in a transverse magnetic field](image)

The goal is to determine eddy currents induced in the conductor by the field and power losses due to them. To achieve it the following simplifications and assumptions were done:
- all conductors are infinitely long,
- the field is independent on \( z \) coordinate,
- the currents in source regions generate time-harmonic magnetic field of angular frequency \( \omega \),
- conductivity, \( \gamma \), and relative permeability, \( \mu_r \), of the considered conductor are constant,
- the reverse reaction on the source conductors can be neglected.

Due to the first two assumptions the problem may be regarded as a 2D one, in which the magnetic vector potential has a \( z \)-component only, which is independent of \( z \) coordinate, i.e.

\[
(1) \quad A = A(r, \phi)I_z, \\
\text{where} \ I_z \ \text{is a unit vector of} \ \phi \text{- polar coordinates.}
\]

The source vector magnetic potential (its \( z \)-component) in the vicinity of the conductor is assumed to have the form

\[
(2) \quad A_0(r, \phi) = B_0 \alpha \sum_{n=1}^{\infty} C_{0n} \left( \frac{r}{a} \right)^n \ \text{trig} \ n \phi, \\
\text{where:} \ B_0 - \text{magnetic flux density,} \ a - \text{conductor’s radius,} \ C_{0n} - \text{dimensionless constants, and}
\]

\[
(3) \quad \text{trig} \ n \phi = c_n \cos n \phi + s_n \sin n \phi, \\
\text{where} \ c_n \ \text{and} \ s_n \ \text{are normalized constants} \ (c_n^2 + s_n^2 = 1). 
\]

Series (2) is assumed to be converge in a ring \( a \leq r < r_0 \).

Magnetic field and eddy currents in the conductor

Inside the cylindrical conductor \( r \leq a \), the \( z \)-component of magnetic vector potential satisfies Helmholtz’s equation

\[
(4) \quad \nabla^2 A_{\text{int}} - k^2 A_{\text{int}} = 0, \\
\text{where}
\]

\[
(5) \quad k^2 = j \omega \mu_0 \gamma, \\
(6) \quad k = \frac{1}{d} + j \frac{1}{\delta}, \quad \delta = \sqrt{\frac{2}{\omega \mu_0 \gamma}},
\]

with \( \delta \) as the skin depth. By using the method of separation of variables the solution of Eq. (4) becomes

\[
(7) \quad A_{\text{int}}(r, \phi) = \sum_{n=0}^{\infty} [C_{n+1}R_n(kr) + D_{n+1}K_n(kr)] \text{trig} n \phi, 
\]
where \( C_0 \) and \( D_0 \) are constants to be found, \( I_r \) and \( K_r \) – modified Bessel functions of order \( n \) of the first and second kind, respectively. To obtain finite values of \( A_{ext} \) for \( r = 0 \) the term with \( K_r \) should be discarded, because when \( r \to 0 \), \( K_r(r) \) tends to infinity. Thus \( D_{ext} = 0 \).

In the vicinity of the conductor, outside the region with source currents, magnetic vector potential (its \( z \)-component) satisfies the Laplace equation

\[
\nabla^2 A_{ext} = 0 .
\]

Its solution has the form

\[
A_{ext} = A_0 + A_1 ,
\]

where \( A_0 \) is the source potential (2), and \( A_1 \) is an adjustment due to the presence of the conductor, equal

\[
A_1 = C_20 + D_20 \ln r + \sum_{n=1}^{\infty} \left( C_{2n} r^n + D_{2n} r^{-n} \right) \text{trig} n \phi ,
\]

where \( C_{2n} \) and \( D_{2n} \) are constants to be determined. \( A_{ext} \) tend to \( A_0 \) for large \( r \), hence \( A_1 \) should vanish then, from which it follows that \( D_{2n} = C_{2n} = 0 \).

Constants \( C_{2n} \) in Eq. (7) and \( D_{2n} \) in (10) are determined by the continuity conditions for electromagnetic field on the cylinder’s surface (\( r = a \)), i.e.

\[
A_{int} = A_{ext} , \quad \frac{1}{\mu_t} \frac{\partial A_{int}}{\partial r} = \frac{\partial A_{ext}}{\partial r} .
\]

The conditions lead to the following equations:

\[
C_{10} \phi_0(\kappa) = C_{20} , \quad \frac{1}{\mu_t} C_{10} \phi_t(\kappa) = 0 ,
\]

for \( n = 0 \), and

\[
C_n \phi_n(\kappa) = B_0 a C_{2n} + D_{2n} a^{-n} , \quad \frac{1}{\mu_t} C_n \phi'_n(\kappa) = n B_0 C_{2n} - D_{2n} n a^{-n-1} ,
\]

for \( n > 0 \), where

\[
\kappa = ka = \frac{a}{\delta} = j a = \alpha + j \alpha , \quad \alpha = \frac{a}{\delta} .
\]

Hence, \( C_{10} = C_{20} = 0 \), and

\[
C_n = B_0 a C_{2n} M_n , \quad D_{2n} = B_0 a C_{2n} (M_n - 1) a^n ,
\]

where

\[
M_n = M_n(\mu_t, \alpha) = \frac{2 \mu_t}{\mu_t + I_n(\kappa)} ,
\]

with an auxiliary function defined as

\[
\overline{I}_n(z) = \frac{z I_n(z)}{n I_n(z)} .
\]

Taking into account the above expressions, the vector magnetic potential can be expressed as:

\[
A_{int} = B_0 a \sum_{n=1}^{\infty} C_{2n} M_n \frac{I_n(\kappa r)}{I_n(\kappa)} \text{trig} n \phi ,
\]

\[
A_{ext} = A_0 + B_0 a \sum_{n=1}^{\infty} C_{2n} (M_n - 1) \left( \frac{a}{r} \right)^n \text{trig} n \phi .
\]

Density of eddy currents induced in the conductor, \( J_{int} \), can be expressed as \( \text{jw} \overline{A}_{int} \), which yields

\[
J_{int} = -\frac{B_0}{\mu_0 \mu_t} \sum_{n=1}^{\infty} C_{2n} M_n \frac{I_n(\kappa r)}{I_n(\kappa)} \text{trig} n \phi .
\]

Magnetic flux density, \( B \), can be found as \( \text{curl} A \), from which it follows that inside the conductor

\[
B_{int} = \frac{B_0}{\mu_0 \mu_t} \sum_{n=1}^{\infty} C_{2n} M_n \frac{a}{r} \frac{I_n(\kappa r)}{I_n(\kappa)} \text{trig} n \phi ,
\]

\[
B_{ext} = \frac{B_0}{\mu_0 \mu_t} \sum_{n=1}^{\infty} C_{2n} M_n \frac{a}{r} \frac{I_n(\kappa r)}{I_n(\kappa)} \text{trig} n \phi .
\]

Similar formulas express the magnetic flux density outside the conductor.

### Joule’s power losses

Apparent power supplied to the conductor can be found as the Poynting vector flux through the surface of the cylindrical conductor. Apparent power \( S \) per unit of length of the conductor can be expressed as

\[
S = \frac{1}{\mu_0 \mu_t} \oint A_{int} \frac{\partial A_{int}}{\partial r} r = a \text{d} \phi .
\]

Putting Eq. (18) into the above formula, performing the integration yields

\[
S = \frac{B_0}{\mu_0 \mu_t} \sum_{n=1}^{\infty} n |C_{2n}|^2 M_n^2 M_n^2 \overline{I}_n(\kappa) .
\]

The real part of this expression determines power losses in the conductor per unit of length. It can be written as

\[
P' = \text{Re} S = \sum_{n=1}^{\infty} |C_{2n}|^2 P_n ,
\]

where

\[
P_n = \frac{B_0^2 \pi}{\gamma_0 \mu_0} p_n(\alpha, \mu_t) ,
\]

\[
p_n(\alpha, \mu_t) = \frac{8 n \alpha^2}{|\mu_t + \overline{I}_n(\alpha + j \alpha)|^2} \text{Im} \left[ \overline{I}_n(\alpha + j \alpha) \right] ,
\]

with \( \alpha = a\delta \) – see Eq. (14). Dimensionless function \( p_n(\alpha, \mu_t) \) is the contribution of \( n \)-th spatial harmonic into Joule’s power losses. Figures 2 and 3 present \( p_n \alpha^2 \) as a function of \( \alpha \) for different values of \( n \) and \( \mu_t \).
Expanding (27) into power series and preserving the term of highest order in \( \alpha \), we obtain

\[
\alpha \approx \frac{f_{0}\mu_{t}}{f_{0}} = \frac{1}{\pi \mu_{t}}.
\]

Frequency \( f_{0} \) is a reference frequency, at which the skin depth \( \delta \) for a non-magnetic cylinder equals the radius \( a \) of the cylinder’s cross-section. Figure 4 shows changes in \( \mu \alpha \) versus frequency \( f \) for different values of \( n \) and \( \mu_{t} \). Power losses for low frequencies are greater in magnetic conductor, but there is a range of frequency in which the contrary relation takes place.

Power losses due to \( n \)-th spatial harmonic at low frequencies are directly proportional to the conductivity of the conductor as well as to the fourth power of radius of the conductor. For non-magnetic cylinder (\( \mu_{t} = 1 \)) the above expression simplifies to

\[
P_{n} \propto 2^{\frac{3}{2}} \pi \frac{a^{4}}{n+1} f^{2}.
\]

while for magnetic (\( \mu_{t} \gg 1 \))

\[
P_{n} \propto 42^{\frac{3}{2}} \pi \frac{a^{4}}{n+1} f^{2} = 4P'_{n} \mu_{t} = 1.
\]

Compare the result obtained with Fig. 4, in which it is visible that \( p_{n} \) at very low frequencies is approximately four times greater for magnetic conductor than for non-magnetic one. It is worth noting that this approximation is quite good even for \( \alpha = 1 \) (i.e. \( f \approx f_{0}/\mu_{t} \)), because the coefficient at \( \alpha \) in the power series expansion of expression (27) is much less than the one at \( \alpha^{4} \). This is noticeable especially for \( n > 1 \) or \( \mu_{t} \gg 1 \).

To obtain approximate formulas for high frequency let us observe that for \( |z| \gg n \) it holds [6]

\[
\overline{f}_{n}(z) \approx \frac{z}{n}.
\]
Thus, for $\alpha \gg n$, which corresponds to $f \gg n^2\omega/\mu_r$, formula (27) can approximated by

$$p_n \approx 8n^2\frac{\alpha^3}{(n\mu_r + \alpha)^2 + \alpha^2}.$$  

(34)

If $\alpha \gg n\mu_r$, what is fulfilled for non-magnetic cylinder (or magnetic for very large $\alpha$), the formula reduces to

$$p_n \approx 4n^2(\alpha - n\mu_r) = 4n^2\left(\frac{\mu_r f}{f_0} - n\mu_r\right).$$

(35)

At $\mu_r = 1$ it gives

$$P'_n = B_0^2\frac{4\pi}{\mu_0} n^2 a \left(\frac{n\pi f}{\mu_0} - n\right).$$

(36)

Power losses due to $n$-th spatial harmonic at high frequencies for non-magnetic cylinder are proportional to the square root of frequency, the square of harmonic number, the radius of the cylinder, and inversely proportional to the square root of conductivity.

Subsequently, if $\alpha \ll n\mu_r$, what can occur only for a magnetic cylinder (because we assumed that $\alpha \gg n$), we obtain

$$p_n \approx 8\frac{\alpha^3}{\mu_r^2} = 8\frac{f^2}{\sqrt{\mu_r} f_0},$$

(37)

$$P'_n \approx 8\frac{n^2}{\mu_0\mu_r} f^2.$$  

(38)

Thus, Joule’s power losses due to $n$-th spatial harmonic for large enough frequency in magnetic cylinder are proportional to the third power of the square root of frequency, the third power of the radius of the cylinder, the square root of conductivity, and inversely proportional to the square root of permeability. It is worth noting that they do not depend on the number of harmonic.

**Numerical example**

Let the source of magnetic field is a long filament carrying time-harmonic current $I$ (see Fig. 6). The distance between axes of filament and the considered cylinder equals $d$.

![Cylinder near a long filament carrying time-harmonic current](image)

Fig. 6. Cylinder near a long filament carrying time-harmonic current

The $z$-component of the source vector magnetic potential at point $P$ equals

$$A_0 = \frac{\mu_0 I}{2\pi} \ln \frac{1}{\rho} + C,$$

(39)

where $\rho$ is the distance between point $P$ and the filament, and $C$ is a constant that can be freely assumed. Expressing distance $\rho$ by $r$, $d$ and $\phi$ yields

$$0.1 \quad 1.0 \quad 1.5 \quad 2.0 \quad 2.5\quad 3.0$$

![Distribution of the maximum value of magnetic flux density on the surface of the cylinder for different values of parameters](image)

Fig. 8. Distribution of the maximum value of magnetic flux density on the surface of the cylinder for different values of parameters: a) $d = 2\alpha$, $\mu_r = 1$, $\alpha = 1$, 3, 5; b) $\alpha = 3$, $\mu_r = 1$, $d = 1.5\alpha$, 2$\alpha$, 5$\alpha$; c) $d = 2\alpha$, $\alpha = 3$, $\mu_r = 1$, 1000; the upper parts refer to the considered configuration, while the lower to the first spatial harmonic ($n = 1$); axes scaled in $\sqrt{2}B_0$. 

\[ \frac{1}{d} \]
Using the following expansion (truthful for $x < 1$) \[7\]

\[
\ln(1 + x^2 - 2x \cos \phi) = -2 \sum_{n=1}^{\infty} \frac{1}{n^4} \cos n \phi ,
\]

leads us to

\[
A_0 = \frac{\mu_0 l}{2\pi} \sum_{n=1}^{\infty} \left( \frac{\ln d}{d} \right)^n \cos n \phi + \left( C - \frac{\mu_0 l}{2\pi} \ln d \right),
\]

which is truthful for $r < d$ (i.e. the convergence radius $r_0 = d$). Comparing the result obtained with Eq. (2) allows us to identify that

\[
B_0 = \frac{\mu_0 l}{2\pi a}, \quad C_{0n} = \frac{1}{n^4} \left( \frac{a}{d} \right)^n, \quad c_n = 1, \quad s_n = 0 .
\]

Figure 7 shows the lines of magnetic flux density and distribution of current density at a certain phase for arbitrary parameters. Figure 8 shows the distribution of the maximum value of magnetic flux density on the surface of the cylinder for different values of parameters. The upper parts of the figures correspond to the considered configuration, while the lower ones – to the uniform field due to the first spatial harmonic ($n = 1$). Figure 9 presents power losses $P'$ as a function of $\alpha$ for different values of $d/a$ and $\mu_r = 1$.

**Concluding remarks**

Based on the above considerations the following conclusions can be formulated:

- for low frequency power losses are inversely proportional to the number of spatial harmonic of the source magnetic field; they are approximately four times greater in a magnetic material than in a non-magnetic one;
- for high frequency power losses are approximately proportional to the square of spatial harmonic number in a non-magnetic cylinder while are independent of the harmonic number in a magnetic one;
- for low frequency power losses are proportional to the square of frequency while for high frequency they are proportional to the square root of frequency in a non-magnetic cylinder and to the third power of the square root of the frequency in a magnetic one (which remains in accordance with known facts).

**REFERENCES**


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