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On fractal dimension estimation

***Abstract**. The paper deals with an algorithm for Hausdorff dimension estimation based on box-counting dimension calculation. The main goal of the paper is to propose a new approach to box-counting dimension calculation with less computational demands.*

Streszczenie.** W artykule opisano algorytm estymacji wymiaru Hausdorffa oparty o wyznaczanie wymiaru pudełkowego. Głównym celem pracy jest zaproponowanie nowego podejścia do wyznaczania wymiaru pudełkowego, które ma znacznie mniejszą złożoność obliczeniową. **(Estymacja wymiaru fraktalnego)

Keywords: fractal dimension, box-counting dimension, Hausdorff dimension, chaotic systems
Słowa kluczowe: wymiar fraktalny, wymiar pudełkowy (pojemnościowy), wymiar Hausdorffa, systemy chaotyczne.

Introduction

There are two ways how to approach Fractal dimensions. In mathematics is the theory of fractals a part of the most theoretical and abstract field – general topology. On the other hand fractal dimensions are used in a number of practical applications. For instance Fractal dimensions are commonly used tools for describing bio-electric signals like EEG, ECG, etc. There are a number of publications which deal with a connection between particular fractal dimension value and specific brain activity. As examples we can mention usage of fractal dimension as a measure of EMG (electromyogram) interference pattern [1], using fractal dimensions for describing end predicting epileptic seizures [2] and for detecting of schizophrenia [3]. Unfortunately algorithms for fractal dimensions calculation usually have high computational demands, especially for high-dimensional systems. The goal of the paper is to propose a modification of box-counting dimension calculation in order to reach lower computational demands.

Hausdorff dimension

Hausdorff dimension is a descriptor of the complexity of geometry of the given set. The set can be trajectory of any dynamical system and can also be reconstructed from measured data.

Suppose that $\mathcal{A}$ is the set whose dimension is to be calculated. Let $C(r,\mathcal{A}) = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k\}$ is a finite cover of the set $\mathcal{A}$ by sets whose diameters are less than r. Then the function

$$\Gamma(\mathcal{A},D,r) = \lim_{r\to 0} \inf_{C(r,\mathcal{A})} \sum_i \boldsymbol{\delta}_i^D \tag{1}$$

defines a measure of the set $\mathcal{A}$. For most values of D, the limit $r \to 0$ leads to a degenerate measure: either $\Gamma \to 0$ or $\Gamma \to \infty$. There is a unique transition point

$$D_H = \inf\{D : \Gamma(\mathcal{A},D)\}$$

that defines the Hausdorff dimension [1], where

$$\Gamma(\mathcal{A},D) = \lim_{r\to 0} \Gamma(\mathcal{A},D,r) = \begin{cases} \infty & \text{for } D < D_{\mathrm{H}} \\ 0 & \text{for } D > D_{\mathrm{H}} \end{cases}. \tag{2}$$

Box-Counting Dimension

However Hausdorff dimension is an important theoretical tool, the definition itself does not offer any guideline to an estimate calculation. The condition that calculation of infimum is provided over all possible covers is a serious problem for numerical implementation [1].

Box-counting dimension is an estimate of Hausdorff dimension based on covering of investigated set by a fixed size grid, with the grid size r. In this case Eq. (1) becomes into form

$$\Gamma(\mathcal{A},D,r) = \sum_i r^D = k(r) r^D , \tag{3}$$

where $k(r)$ is a number of grid boxes which contain any part of investigated set. The box-counting dimension is the value of D on the transition between $\Gamma \to 0$ and $\Gamma \to \infty$. The estimate of the Hausdorff dimension can be written in the form [3]

$$D_H \approx \lim_{r\to 0} \frac{\log[1/k(r)]}{\log r} . \tag{4}$$

The problem of implementing Eq. (4) is very (logarithmic) slow convergence. Instead of using this expression, it is usually calculated box-counting dimension estimate based on equation

$$D_H \approx \frac{-\Delta[\log k(r)]}{\Delta[\log(r)]} , \tag{5}$$

with sufficiently small r. The problem is the determining if given box of grid contains a point (or points) of trajectory over all boxes in grids. The goal of the paper is to propose an algorithm based on rounding and integer arithmetic.

Implementation

The general problem of fractal dimension algorithms is the fact that computational demands increase very fast with topological dimension. Suppose that the investigated set is located in n-dimensional cube. Let the length of each edge of the cube is D and each edge is divided into N parts. Then the whole cube is divided into N^D boxes. It is necessary for each box decide if it contains a part of investigated set or not. Therefore the computational demands increase exponentially with exponent D.

The paper deals with the algorithm for calculation of box-counting dimension from measured data. It means that the investigated set consists of discrete points. The set can be relatively sparse depending on sampling frequency. On the other hand we have a-priory information that these points are parts of continuous trajectory in the state space.

The proposed algorithm is a modification of standard box-counting dimension algorithm.

The space of topological dimension n is normalized and divided into N^D boxes with length of edge r and then we search for all boxes which contain any part of trajectory. That means all boxes containing at least one point and all boxes between any two following points (suppose linear approximation of measured data).

The problem of localization of trajectory can by understood as investigation of spatial discretization. In practical cases the trajectory reconstructed from measured data consists always of a sequence of quantized (integer) numbers. The points of trajectory are assigned to appropriate box of given space only by rounding off with given precision.

The spatial resolution can be relatively low. Suppose that data were sampled with frequency f_s =1 ms for 40 seconds, that means we have 40.000 points in the state space. If each axis is divided into 100 parts, the whole space is divides into 10^6 boxes, with finer resolution the number of adjacent boxed decrease and an influence of linear approximation increase. Paradoxically finer resolution worsens the Hausdorff dimension estimate.

Because of relatively rough spatial resolution it is possible to use a special vector $P(x_1, x_2, x_3 ...)$ that contains the whole information about given point position in state space. The construction of the vector P for the three-dimensional case is illustrated by expression

$$P(x, y, z) = x_1 << 16 + x_2 << 8 + x_3, \tag{6}$$

where x_1, x_2, x_3 are state variables (integer time series) reconstructed from measured data and "<<" stands for bit shift. This vector can be effectively sorted by any standard methods. After eliminating of duplicity values we can easily get the number of non-empty boxes. The box-counting dimension is then calculated by Eq. (5).

In practical applications is more important the ability of real time processing than accuracy of Hausdorff dimension estimate. In number of application it is even sufficient distinguishing if measured signal is periodic or quasi-periodic, stochastic or chaotic.

Example

The well known nonlinear system with chaotic behaviour is the Lorenz system described by set of non-linear differential equations

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1), \\ \dot{x}_2 &= x_1(r - x_3) - x_2, \\ \dot{x}_3 &= x_1 x_2 - b x_3. \end{aligned} \tag{7}$$

It is well-known that for values of parameters $\sigma = 10, b = 8/3$ and $r = 28$ (σ is called the Prandtl number and r is called Rayleigh number) generates the Lorenz system chaotic behaviour, which can be recognized by value of Hausdorff dimension. The Hausdorff dimension of trajectory of Lorenz system was determined to be D_H = 2.06±0.01 (Peter Grassberger, 1983).

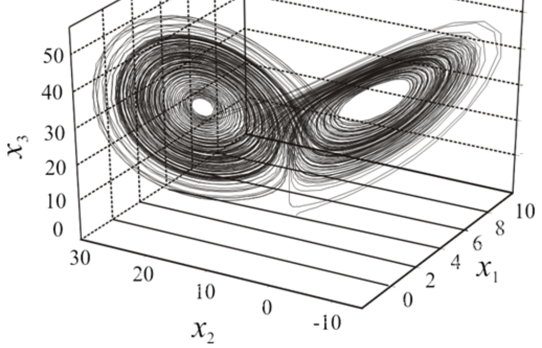

Fig. 1: Trajectory of the Lorenz system in the state space

Trajectory of Lorenz system is depicted in the Fig. 1. The set consisting of trajectory of the Lorenz system is so called strange attractor. The time evaluation of state variables shows Fig. 2. It can be noticed that state variable waveforms are not periodical however they seems to be regular.

The state variables waveforms can be discretized by the technique described above. The trajectory of the Lorenz system in discretized space is depicted in the Fig. 3. However the sampling frequency was hundred times higher than the highest frequency in the spectrum the points forming the trajectory are still sparsely distributed in space. The Fig. 4 shows the dependence of Hausdorff dimension estimate on length of observation. Relatively slow convergence is a general problem of algorithms based on box-counting dimension calculation.

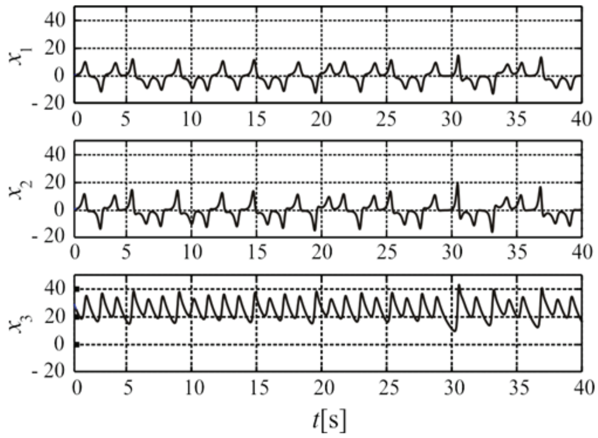

Fig. 2: Time evolution of the state variables of the Lorenz system

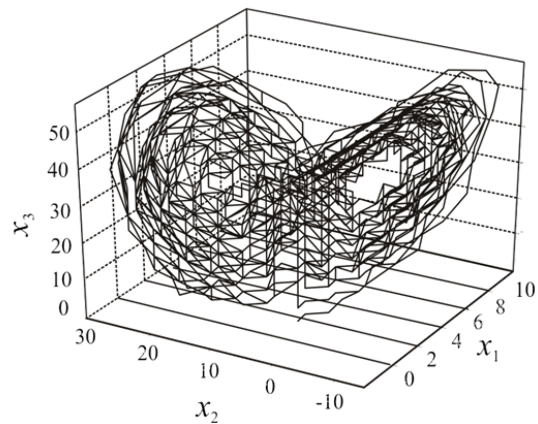

Fig. 3: Quantized trajectory of the Lorenz system.

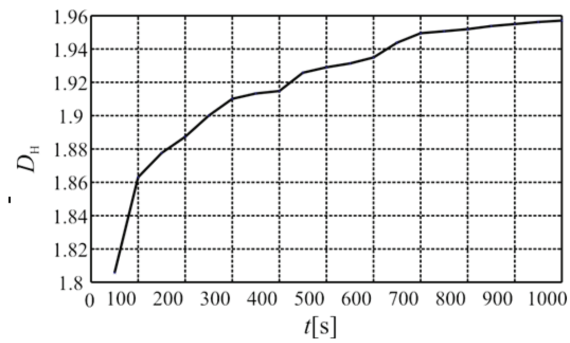

Fig. 4: Dependence of the value of the Hausdorff dimension on the length of observation

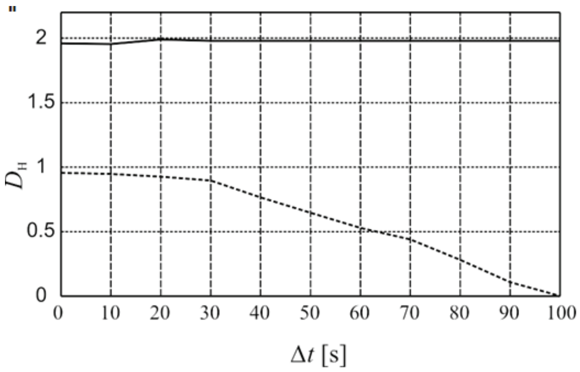

Fig. 5: Hausdorff dimension estimates for chaotic (solid line) and dissipative (dashed line) Lorenz system.

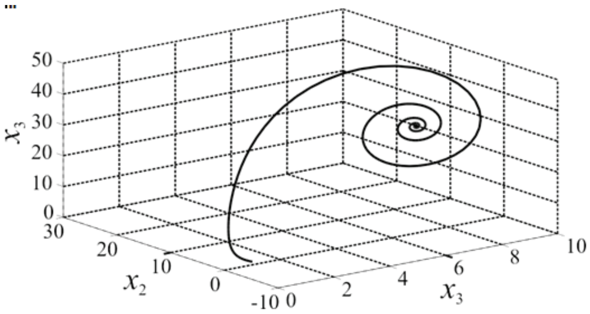

Fig. 6: Trajectory of dissipative Lorenz system

The picture in Fig. 5. shows the dependence of Hausdorff dimension estimate on the length of observation for two cases, the first one is mentioned chaotic Lorenz system and the second one is dissipative Lorenz system (σ = 1). It is obvious that for dissipative case decrease the Hausdorff dimension estimate to zero. The trajectory of dissipative Lorenz system is depicted in Fig. 6.

Conclusions

The proposed algorithm of the fractal dimension estimate calculation belongs to the class of box-counting dimensions algorithms. Its advantages are lower computational demands compared with classical box-counting dimension algorithm and a possibility to implement this algorithm by integer arithmetic allowing realization with signal processors with integer arithmetic or programmable logical circuits. The proposed algorithm is applicable for relatively fast estimate of Hausdorff dimension, but as well as other box-counting dimension algorithms shows slow convergence to precise value of Hausdorff dimension.

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REFERENCES

[1] Theiler, J.: Estimating fractal dimension, *Journal Optical Society of America*, Volume 7, Issue 2, June 1990, Pages 1055-1073.

[2] E.T. Bullmore, M.J. Brammer, P. Bourlon, G. Alarcon, C.E. Polkey, R. Elwes, C.D. Binnie: Fractal analysis of electroencephalographic signals intracerebrally recorded during 35 epileptic seizures: evaluation of a new method for synoptic visualisation of ictal events, *Electroencephalo-graphy and Clinical Neurophysiology*, Volume 91, Issue 5, November 1994, Pages 337-345.

[3] A. L. Sandu, I. A. Rasmussen Jr., A. Lundervold, F. Kreuder, G. Neckelmann, K. Hugdahl, K. Specht: Fractal dimension analysis of MR images reveals grey matter structure irregularities in schizophrenia, *Computerized Medical Imaging and Graphics*, Volume 32, Issue 2, March 2008, Pages 150-158.

[4] J. Theiler: Estimating fractal dimension, *Journal Optical Society of America*, Volume 7, Issue 2, June 1990, Pages 1055-1073.

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