Bartosz SAWIK

AGH University of Science & Technology

Bi-Criteria Portfolio Optimization Models with Percentile and Symmetric Risk Measures by Mathematical Programming

Abstract. *The purpose of this paper is to compare three different bi-criteria portfolio optimization models. The first model is constructed with the use of percentile risk measure Value-at-Risk and solved by mixed integer programming. The second one is constructed with the use of percentile risk measure Conditional Value-at-Risk and solved by linear programming. The third one is constructed with the use of a symmetric measure of risk - variance of return - as in the Markowitz portfolio and solved by quadratic programming. Computational experiments are conducted for bi-criteria portfolio stock exchange investments. The results obtained prove, that the bi-objective portfolio optimization models with Value-at-Risk and Conditional Value-at-Risk could be used to shape the distribution of portfolio returns. The decision maker can assess the value of portfolio return and the risk level, and can decide how to invest in a real life situation comparing with ideal (optimal) portfolio solutions. The proposed scenario-based portfolio optimization problems under uncertainty, formulated as a bi-objective linear, mixed integer or quadratic program are solved using commercially available software (AMPL/CPLEX) for mathematical programming.*

Streszczenie. *W pracy zaprezentowano problem portfelowy, dla którego zaproponowane zostały trzy dwukryterialne modele optymalizacji. W każdym z zaimplementowanych modeli funkcja celu jest ważona suma dwóch kryteriów, które podlegają optymalizacji dla konkretnego portfela. W pierwszym modelu z warunkową wartością zagrożoną zwrotu (Conditional Value-at-Risk) kryteria decyzyjne to maksymalizacja CVaR i przewidywanego zwrotu portfela. W drugim modelu z wartością zagrożoną zwrotu (Value-at-Risk) minimalizowane jest prawdopodobienstwo ryzyka straty oraz maksymalizacja oczekiwanego zwrotu z portfela. Trzeci model to omawiany szeroko w literaturze model portfela Markowitza zmodyfikowany tak, by funkcja kryterialna była ważona sumą kryteriów i minimalizowała ryzyko portfela zdefiniowanego jako macierz kowariancji historycznych zwrotów oraz maksymalizowała przewidywany zwrot portfela. Modele te zostały zaimplementowane: w pierwszym - używając metody programowania mieszanego całkowitoliczbowego, a w drugi i trzeci z użyciem programowania liniowego i kwadratowego. Efektywność zaproponowanych modeli została zweryfikowana eksperymentalnie, zwracając szczególną uwagę na czas obliczeń oraz przewidywany zwrot portfela. Zamieszczono wyniki eksperymentów obliczeniowych przeprowadzonych z zastosowaniem optymizatora CPLEX i jezyka modelowania algebraicznego AMPL.* ***(Zaimplementowane metodami programowania matematycznego dwukryterialne modele optymalizacji portfelowej z percentylowymi oraz symetrycznymi miarami ryzyka)***

Keywords: multi-criteria decision making, portfolio optimization, percentile and symmetric risk measures, weighting approach, mathematical programming

Słowa kluczowe: wielokryterialne podejmowanie decyzji, optymalizacja portfelowa, percentylowe i symetryczne miary ryzyka, ważona funkcja celu, programowanie matematyczne

Introduction

The development of new techniques in operational research, as well as the progress in computer and information technologies, has given rise to new approaches for modeling the problem for portfolio selection. The multi-criteria decision making provides a solid methodological basis for resolving the inherent multi-criteria nature of the problem. The multi-dimensional nature of the portfolio selection problem has been emphasized by many researchers, from the fields of financial engineering and multi-criteria decision making.

The portfolio problem involves computing the proportion of the initial budget that should be allocated in the available securities, is at the core of the field of financial management. A fundamental answer to this problem was given by Markowitz ([1]) who proposed the mean-variance model which laid the basis of modern portfolio theory. Since the mean-variance theory of Markowitz, an enormous amount of papers have been published extending or modifying the basic model in three directions (e.g. [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]).

Although the original Markowitz model forms a quadratic programming problem, many attempts have been made to linearize the portfolio optimization procedure (for instance [5], [16]). The linear program solvability is very important for applications to real-life financial and other decisions where the constructed portfolios have to meet numerous side constraints.

The proposed bi-criteria portfolio approach allows two percentile measures of risk in financial engineering: value-at-risk (VaR) and conditional value-at-risk (CVaR) to be applied for managing the risk of portfolio loss. The proposed linear programming model provide the decision maker with a simple tool for evaluating the relationship between expected and worst-case loss of portfolio return.

Selected Measures of Risk

Value-at-Risk (VaR) represents the minimum return (maximal loss accepted by a decision maker; for instance VaR = -1 – minus one percent) associated with a specified confidence level of outcomes (i.e. the likelihood that a given portfolio's return will not be less than the amount defined as VaR).

However, VaR does not account for properties of the return distribution beyond the confidence level and hence does not explain the magnitude of the return when the VaR limit is exceeded.

On the other hand, CVaR (Conditional Value-at-Risk) focuses on the tail of the return distribution, that is, on outcomes with the lowest return.

Since VaR and CVaR measure different parts of the return distribution, VaR may be better for optimizing portfolios when good models for tails are not available, otherwise CVaR may be preferred, e.g. ([17], [18], [19]).

When using CVaR to maximize worst-case return (minimizing maximal accepted portfolio loss), CVaR is always less than VaR. On the other hand, VaR is a better choice to measure the risk of critical portfolio returns.

VaR and CVaR have been widely used in financial engineering in the field of portfolio management (e.g. [18]). CVaR is used in conjunction with VaR and is applied for estimating the risk with non-symmetric cost distributions. Urya-sev ([19]) and Rockafellar and Uryasev ([20], [17]) introduced a new approach to select a portfolio with the reduced risk of high losses. The portfolio is optimized by calculating VaR and minimizing CVaR simultaneously.

Let $\alpha \in (0,1)$ be the confidence level. The percentile measures of risk, VaR and $CVaR$ can be defined as below:

- Value-at-Risk (VaR) at a $100\alpha\%$ confidence level is the targeted return of the portfolio such that for $100\alpha\%$ of outcomes, the return will not be lower than VaR. In other words, VaR is a decision variable based on the α-percentile of return, i.e., in $100(1-\alpha)\%$ of outcomes, the return may not attain VaR.

- Conditional Value-at-Risk ($CVaR$) at a $100\alpha\%$ confidence level is the expected return of the portfolio in the worst $100(1-\alpha)\%$ of the cases. Allowing $100(1-\alpha)\%$ of the outcomes not exceed VaR, and the mean value of these outcomes is represented by $CVaR$.

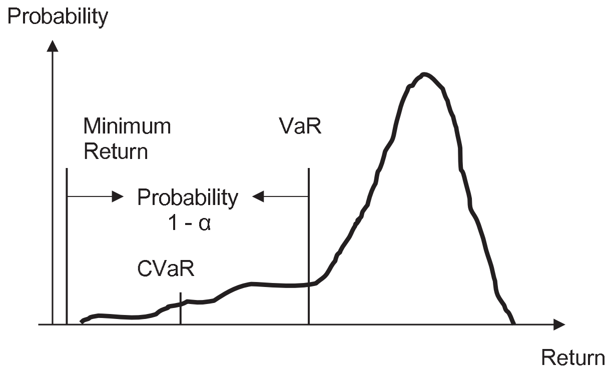

Fig. 1. Value-at-Risk and Conditional Value-at-Risk

Figure 1. illustrates Value-at-Risk and Conditional Value-at-Risk for a given portfolio and the confidence level α.

Portfolio Problems Formulation

The non-dominated solution set of bi-objective mixed integer, linear or quadratic program models *M* can be partially determined by the parametrization on λ of the following weighted-sum program.

Model M_λ

Maximization or minimization $\sum_{l=1}^{m} \lambda_\iota f_\iota$

subject to some specific model constraints (As it is formulated in models presented in this paper.), where $\lambda_1 > \lambda_2 > ... > \lambda_m, \lambda_1 + \lambda_2 + ... + \lambda_m = 1$.

It is well known, however, that the non-dominated solution set of a bi-objective mixed integer or linear or quadratic program such as M_λ cannot by fully determined even if the complete parametrization on λ is attempted (e.g. [21]). To compute unsupported non-dominated solutions, some upper bounds on the objective functions should be added to M_λ (e.g. [22]).

This section includes bi-objective portfolio models. Weighting approach for objective functions has been implemented. The first objective defines risk of portfolio venture, this objective minimizes risk subject to specific constraints. The second objective function maximizes portfolio expected return. First bi-objective portfolio model **M1** is constructed with implementation of conditional value-at-risk as a main risk measure. Second portfolio model **M2** has value-at-risk as a basic measure of risk. Third one **M3** includes modified classical Markowitz portfolio bi-objective model, which is added for results comparison between presented portfolio approaches.

Notations for bi-objective portfolio problems **M1**, **M2**, **M3** formulation are presented in Tables 1 and 2.

Conditional Value-at-Risk Bi-Criteria Portfolio Model

The proposed model **M1** provides a decision maker with a tool for evaluating the relationship between expected and worst-case returns. The portfolio problem presented ([6], [7], [9], [10]) below provides flexibility in how a decision maker wants to balance his/hers risk tolerance with the expected portfolio returns. The bi-criteria weighted-sum portfolio problem consists of two objective functions (1). The first objective is to maximize Conditional Value-at-Risk ($CVaR$) and the second objective is to maximize portfolio expected return.

Table 1. Notations for mathematical models **M1**, **M2**, **M3**

		Indices
i	=	historical time period, $i \in I = \{1, \ldots, m\}$ (i.e. day, week, month, etc.)
j	=	security, $j \in J = \{1, \ldots, n\}$
		Input Parameters
α	=	input parameter in selected problems - confidence level. The mathematical model, where α is the input parameter, is **M1**.
λ	=	weights in the objective functions
$cov(r_i, r_j)$	=	matrix of covariance - the input parameter in the mathematical model **M3**.
p_i	=	probability assigned to the occurrence of past realization i
r_{ij}	=	observed return of jth stock in ith time period
r^{Min}	=	minimum return observed in the market
VaR	=	return Value-at-Risk. The mathematical model, where VaR is the input parameter, is **M2**.

Table 2. Decision Variables for mathematical models **M1**, **M2**, **M3**

		Decision Variables
α	=	variable in a selected problem - confidence level. The mathematical model, where α is the decision variable, is **M2**.
R_i	=	tail return, i.e. the amount by which VaR exceeds return in scenario i
VaR	=	Value-at-Risk of portfolio return based on the α - percentile of return, i.e., in $100\alpha\%$ of historical portfolio realization, the outcome must be greater than VaR. The mathematical model, where VaR is the decision variable, is **M1**.
x_j	=	amount of capital invested in security j
y_i	=	1 if return of portfolio in ith time period is over threshold VaR, 0 otherwise

Model M1

Maximize

(1)
$$\lambda(VaR - (1-\alpha)^{-1}\sum_{i=1}^{m} p_i R_i) + (1-\lambda)(\sum_{i=1}^{m} p_i(\sum_{j=1}^{n} r_{ij} x_j)$$

subject to

$$\sum_{j=1}^{n} x_j = 1 \quad (2)$$

Constraint (2) ensures that all capital is invested in the portfolio (the selected securities).

$$R_i \geq VaR - \sum_{j=1}^{n} r_{ij} x_j; i \in M \quad (3)$$

Risk constraint (3) defines the tail return for scenario i.

$$x_j = 0; j \in N : \sum_{i=1}^{m} p_i r_{ij} \le 0 \tag{4}$$

Constraint (4) eliminates from selection the assets with non-positive expected return over all scenarios.

$$x_j \ge 0; j \in N \tag{5}$$

$$R_i \ge 0; i \in M \tag{6}$$

Equations (5) and (6) are variables of non-negativity conditions.

Variable VaR has no boundary - it is between minus and plus infinity.

A risk-aversive decision maker wants to maximize the Conditional Value-at-Risk, as shown in the objective functions of model **M1**. Note that as R_i is constrained of being positive, the models try to increase VaR and hence positively impact the objective functions. However, large increases in VaR may result in more historic portfolios (scenarios) with tail returns, counterbalancing this effect.

Value-at-Risk Bi-Objective Portfolio Model

The portfolio selection problem could be considered as a single period model of investment. The problem objective is to allocate wealth in different assets to maximize the weighted difference of the portfolio expected return, the threshold of the probability that the return is not less than a required level and the amount of wealth to be invested.

In the proposed approach the allocation quality is measured by the weighted difference of the portfolio expected return and the threshold of the probability that the return is not less than a the required level of VaR (Value-at-Risk).

The model **M2** is formulated as the bi-objective portfolio optimization model with weighted objective function ([23], [15]), i.e. the minimization of the risk probability of portfolio loss versus the maximization of the expected portfolio return. The bi-criteria weighted-sum portfolio problem consists of two objective functions (7). The first objective is to maximize confidence level α and the second objective is to maximize portfolio expected return.

Model M2
Maximize

$$\lambda\alpha + (1-\lambda)(\sum_{i=1}^{m} p_i \sum_{j=1}^{n} r_{ij} x_j) \tag{7}$$

subject to (2), (4), (5) and

$$y_i \le \frac{\sum_{j=1}^{n} r_{ij} x_j - r^{Min}}{VaR - r^{Min}}; i \in M \tag{8}$$

$$y_i \ge \frac{\sum_{j=1}^{n} r_{ij} x_j - r^{Min}}{VaR - r^{Min}} - 1; i \in M \tag{9}$$

$$\sum_{i=1}^{m} p_i(1-y_i) \le 1-\alpha \tag{10}$$

Constraints (8), (9) and (10) prevent the choice of portfolios whose VaR is below the fixed threshold. Whenever the expected portfolio return is below VaR, then y_i must be equal to 0 and $1 - y_i = 1$ in constraint (10). Therefore, all probabilities of event i whose returns are below the VaR threshold are summed up. If the result is greater than $1-\alpha$, then the portfolio is not feasible.

$$y_i \in \{0,1\}; i \in M \tag{11}$$

Constraint (11) defines binary variable y_i. Equal 1 if return of portfolio in time period i is over threshold VaR and 0 otherwise.

$$0 \le \alpha \le 1 \tag{12}$$

Constraint (12) defines confidence level α, as decision variable.

The combination of continuous variable x_j and α and binary variable y_i leads this mixed integer programming NP-hard problem ([24], [25], [26]). If the number of historical observation m is bounded by a constant, there are 2^m ways of fixing the variables y_i.

Bi-Objective Modified Markowitz Portfolio Model

The bi-criteria weighted-sum modified Markowitz portfolio model **M3** consists of two objective functions (13). The first objective is to minimize the portfolio risk defined by the covariance matrix of historical stocks returns, versus the second objective, which is defined as the maximization of the portfolio expected return.

Model M3
Maximize

(13)
$$-\lambda(\sum_{i=1}^{m}\sum_{j=1}^{n} x_i x_j cov(r_i, r_j)) + (1-\lambda)(\sum_{i=1}^{m} p_i(\sum_{j=1}^{n} r_{ij} x_j)$$

subject to (2), (4) and (5).

Computational Results

In this section the strength of presented portfolios approach are demonstrated on computational examples (Fig. Fig. 2– Fig. 4).

In the computational experiments the five levels of the confidence level was applied $\alpha \in \{0.99, 0.95, 0.90, 0.75, 0.50\}$, and for the weighted-sum program the subset of non-dominated solutions were computed by parametrization on λ.

Figure 2. presents comparison of portfolio expected return (**M1**, **M2**, **M3**) for different λ and for 500 historical quotations.

Figure 3. presents comparison of computational times of solved portfolios (**M1**, **M2**, **M3**) for different λ and for 500 historical quotations.

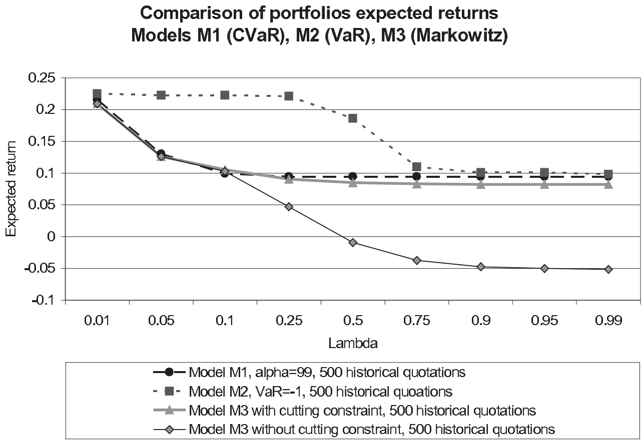

Fig. 2. Comparison of portfolio expected return (**M1**, **M2**, **M3**) for different λ - 500 historical quotations

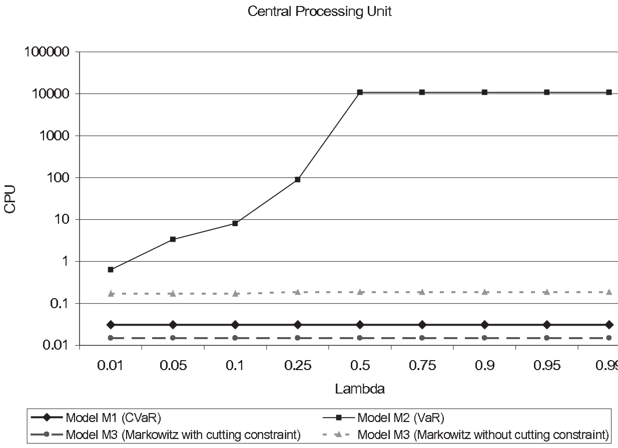

Fig. 3. Comparison of computational times of solved portfolios (**M1**, **M2**, **M3**) for different λ - 500 historical quotations

Figure 4. presents comparison of number of stocks (securities) in portfolios (**M1**, **M2**, **M3**) for different λ and for 500 historical quotations.

All computational experiments were conducted on a laptop with Intel Core 2 Duo T9300 processor running at 2.5GHz and with 4GB RAM. For the implementation of the multi-objective portfolio models, the AMPL programming language and the CPLEX v.11 solver (with the default settings) were applied.

Conclusions

The research problem was to find the relation between the optimization results with Value-at-Risk solved by mixed integer programming and the results of optimization obtained with the use of linear and quadratic programming portfolio models (Conditional Value-at-Risk, Markowitz).

Computational experiments have been conducted for bi-criteria portfolio models of stock exchange investments. The number of selected securities for input data varies from 46 to 240 assets. The historical stocks quotations came from the period from March 10th, 1997 to February 2nd, 2009. This time period includes data from the increase of stock exchange quotations, as well as the economic crisis period. The considered number of data in historical time series is from 500 to 3000 days with assets quoted each day in the whole historical horizon. The portfolios were optimized in an increased time window, which was helpful in evaluating the results of optimization (time-varying optimal portfolio).

The bi-criteria portfolio optimization models with Conditional Value-at-Risk ($CVaR$) as a risk measure can be used

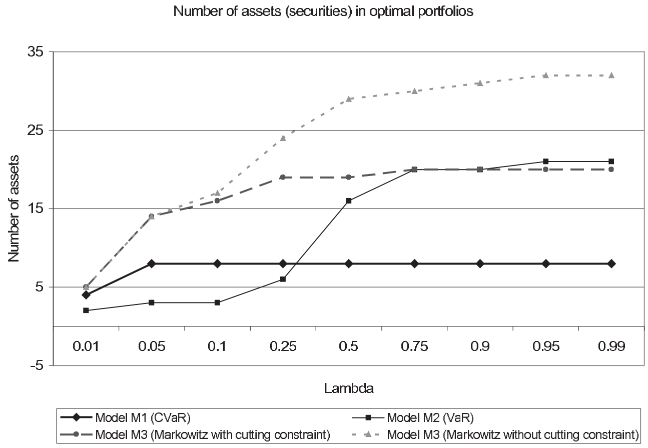

Fig. 4. Comparison of number of stocks (securities) in portfolios (**M1**, **M2**, **M3**) for different λ - 500 historical quotations

to support on-line stock market investments, since the computational times required to find the optimal solution is relatively short, regardless of the size of the input data.

The presented models provide a decision maker with a tool for evaluating the relationship between expected and worst-case returns. The results obtained from computational experiments proved, that multi-objective portfolio optimization models with Value-at-Risk (VaR) and Conditional Value-at-Risk ($CVaR$) could be used to shape the distribution of portfolio returns in a favorable way for a decision maker. The decision maker can assess the value of portfolio return and the risk level, and can decide how to invest in a real life situation comparing with ideal (optimal) portfolio solutions.

The portfolios obtained with both methods (mixed-integer or linear programming) were often similar, which have shown their capability of solving the corresponding problems. It means that a suboptimal portfolio for the integer program with Value-at-Risk (VaR) as optimality criterion can be found by solving the corresponding linear program for the portfolio problem with Conditional Value-at-Risk ($CVaR$) as an optimality criterion. The proposed scenario-based portfolio optimization problems under uncertainty, formulated as a bi-objective linear, mixed integer and quadratic program were solved using commercially available software (AMPL/CPLEX) for mathematical programming.

The nature of the portfolio problem focuses on a compromise between the construction of objectives, constraints and decision variables in a portfolio and the problem complexity of the implemented mathematical models. There is always a trade off between computational time and the size of an input data, as well as the type of mathematical programming formulation (linear or mixed integer).

Acknowledgement

The author is grateful to two anonymous reviewers for their comments. This work has been partially supported by Polish Ministry of Science & Higher Education (MNISW) grant # 6459/B/T02/2011/40 and by AGH.

REFERENCES

[1] Markowitz H.M.: Portfolio selection, *Journal of Finance*, 7, pp. 77–91, 1952.

[2] Alexander G.J., Baptista A.M.: A Comparison of VaR and CVaR Constraints on Portfolio Selection with the Mean-Variance Model, *Management Science*, 50(9), pp. 1261–1273, 2004.

[3] Benati S., Rizzi R.: A mixed integer linear programming formulation of the optimal mean/Value-at-Risk portfolio problem, *European Journal of Operational Research*, 176, pp. 423–434, 2007.

[4] Mansini R., Ogryczak W., Speranza M.G.: Conditional value at risk and related linear programming models for portfolio optimization, *Annals of Operations Research*, 152, pp. 227–256, 2007.
[5] Ogryczak W.: Multiple criteria linear programming model for portfolio selection, *Annals of Operations Research*, 97, pp. 143–162, 2000.
[6] Sawik B.: Downside Risk Approach for Multi-Objective Portfolio Optimization, In: Klatte D., Lüthi H.-J., Schmedders K. (Eds.), *Operations Research Proceedings 2011*, Operations Research Proceedings, Springer-Verlag, Berlin, Heidelberg, Germany, pp. 191–196, 2012.
[7] Sawik B.: Conditional Value-at-Risk vs. Value-at-Risk to Multi-Objective Portfolio Optimization, In: Lawrence K.D., Kleinman G. (Eds.), *Applications of Management Science, Volume 15*, Emerald Group Publishing Limited, Bingley, UK, Vol. 15, Ch. 13, pp. 277–305, 2012.
[8] Sawik B.: A Review of Multi-Criteria Portfolio Optimization by Mathematical Programming, In: Dash G.H., Thomaidis N. (Eds.), *Recent Advances in Computational Finance*, Nova Science Publishers, New York, USA, (forthcomming 2012).
[9] Sawik B.: Conditional value-at-risk and value-at-risk for portfolio optimization model with weighting approach, *Automatyka*, AGH, 15(2), pp. 429–434, 2011.
[10] Sawik B.: A Bi-Objective Portfolio Optimization with Conditional Value-at-Risk, *Decision Making in Manufacturing and Services*, 4(1-2), pp. 47–69, 2010.
[11] Sawik B.: Selected Multi-Objective Methods for Multi-Period Portfolio Optimization by Mixed Integer Programming, In: Lawrence K.D., Kleinman G. (Eds.), *Applications of Management Science, Volume 14, Applications in Multi-Criteria Decision Making, Data Envelopment Analysis and Finance*, Emerald Group Publishing Limited, Bingley, UK, Vol. 14, Ch. 1, pp. 3–34, 2010.
[12] Sawik B.: A Reference Point Approach to Bi-Objective Dynamic Portfolio Optimization, *Decision Making in Manufacturing and Services*, 3(1-2), pp. 73–85, 2009.
[13] Sawik B.: Lexicographic and Weighting Approach to Multi-Criteria Portfolio Optimization by Mixed Integer Programming, In: Lawrence K.D., Kleinman G. (Eds.), *Applications of Management Science, Volume 13, Financial Modeling Applications and Data Envelopment Applications*, Emerald Group Publishing Limited, Bingley, UK, Vol. 13, Ch. 1, pp. 3–18, 2009.
[14] Sawik B.: A weighted-sum mixed integer program for bi-objective dynamic portfolio optimization, *Automatyka*, AGH, 13(2), pp. 563–571, 2009.
[15] Sawik B.: A Three Stage Lexicographic Approach for Multi-Criteria Portfolio Optimization by Mixed Integer Programming, *Przegląd Elektrotechniczny (Electrical Review)*, 84(9), pp. 108–112, 2008.
[16] Speranza M.G.: Linear programming models for portfolio optimization, *Finance*, 14, pp. 107–123, 1993.
[17] Rockafellar R.T., Uryasev S.: Conditional value-at-risk for general loss distributions, *The Journal of Banking and Finance*, 26, pp. 1443–1471, 2002.
[18] Sarykalin S., Serraino G., Uryasev S.: Value-at-Risk vs. Conditional Value-at-Risk in Risk Management and Optimization, In: Chen Z-L., Raghavan S., Gray P. (Eds.), *Tutorials in Operations Research, INFORMS Annual Meeting, Washington D.C., USA, October 12–15*, 2008.
[19] Uryasev S.: Conditional value-at-risk: optimization algorithms and applications, *Financial Engineering News*, Issue 14, February 2000.
[20] Rockafellar R.T., Uryasev S.: Optimization of conditional value-at-risk, *The Journal of Risk*, 2(3), pp. 21–41, 2000.
[21] Steuer R.E.: Multiple Criteria Optimization: Theory, Computation and Application, A John Wiley & Sons, New York, USA, 1986.
[22] Alves M.J., Climaco J.: A review of interactive methods for multiobjective integer and mixed-integer programming, *European Journal of Operational Research*, 180, pp. 99–115, 2007.
[23] Sawik B.: Weighting vs. Lexicographic Approach to Multi-Objective Portfolio Optimization by Mixed Integer Programming, In: Kochan E. (Ed.), *Problems of Mechanical Engineering and Robotics*, Monographic of Faculty of Mechanical Engineering and Robotics, AGH, 36, pp. 201–208, 2007.
[24] Crescenzi P., Kann V. (Eds.): A compendium of NP optimization problems, [web page] `http://www.nada.kth.se/~viggo/wwwcompendium/`, 2005.
[25] Merris R.: *Combinatorics*, Wiley-Interscience, A John Wiley & Sons, Inc., Publication, Hoboken, New Jersey, USA, 2003.
[26] Nemhauser G.L., Wolsey L.A.: *Integer and Combinatorial Optimization*, A John Wiley & Sons, Toronto, Canada, 1999.

***Author**: Bartosz Sawik Ph.D., Eng.*
Department of Applied Computer Science
Faculty of Management
AGH University of Science & Technology
Al. Mickiewicza 30, 30-059 Kraków, POLAND
e-mail: B_Sawik@yahoo.com, BSawik@zarz.agh.edu.pl