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# Ambiguity of Fuzzy Quantities and a New Proposal for their Ranking

***Abstract**. We deal with the problem of evaluating and ranking fuzzy quantities. We call fuzzy quantity any non-normal and non-convex fuzzy set, defined as the union of two, or more, generalized fuzzy numbers. For this purpose we suggest an evaluation defined by a pair index based on "value" & "ambiguity". Either value or ambiguity depend on two parameters connected the first with the optimistic/pessimistic point of view of the decision maker and the second on an additive measure that can be used to express the decision maker's preferences.*

***Streszczenie.** W artykule zaproponowano nową metodę oceny wartości rozmytych (fuzzy). Proponuje się parę oznaczeń – wartość i noiejednoznaczność. (**Niejednoznaczność wartości rozmytych i nowa propozycja ich rankingu**)*

**Keywords:** fuzzy sets, generalized fuzzy numbers, evaluation function, ranking, defuzzication, ambiguity.
**Słowa kluczowe:** wartości rozmyte, logika rozmyta

## 1. Introduction

Either in many fuzzy optimization or in decision making problems, evaluation and/or ranking definitions of fuzzy numbers play an important role. Several proposals of different kind have appeared in literature (see, e.g., [1-4]). Following the line of "utility function" definition in decision making problems one wide group of them proposes to define a real function on the fuzzy numbers set to obtain a real value associated to the fuzzy set useful for its evaluation and ranking too. This approach has produced several proposals with different characteristics. Some of them have chosen to obtain a value into the support of fuzzy set. This is the idea we have decided to follow even in the field of fuzzy quantities. A fuzzy quantity is a fuzzy set obtained by the union of two or more fuzzy numbers not necessarily normal, called generalized fuzzy numbers. These complex sets are usually non–convex and non–normal fuzzy sets. Our choice is due even by the fact that these types of fuzzy sets are the typical output of inference fuzzy systems and the necessity of a way to produce a "defuzzification method" (that is the transformation into a crisp number to obtain the final system output). This idea, like any others in literature, produces equivalent classes very wide, so, to reduce their size, we propose a new definition that uses a lexicographic order based on two index, value and ambiguity. The "value" definition we use is proposed in [5] where the authors present a definition of evaluation of a fuzzy quantity based on α-cut levels and depending on two parameters: a real number connected with the optimistic/pessimistic point of view of the decision maker and an additive measure that allows the decision maker to attribute different weights to each level, according to his preference.

In this paper we add a notion of ambiguity of a fuzzy quantity. Ambiguity is a measure of the vagueness, that is the lack of precision in determining the exact value of a magnitude. Index of ambiguity was suggested for fuzzy numbers to characterize the global spread of the membership function of a fuzzy number [6]. We provide some numerical examples to illustrate the applicability of the proposed method.

In Section 2 we give basic definitions and notations. Furthermore, we present a review of the fuzzy quantities evaluation introduced in [5,7]. In Section 3 we propose a definition of ambiguity of a fuzzy quantity and provide some results. In Section 4 we propose a ranking method for fuzzy quantities based on the value-ambiguity pair and discuss some of its properties. Some numerical examples illustrate our method.

## 2. Definition of fuzzy quantity and its evaluation

Let $X$ denote a universe of discourse. A fuzzy set $A$ in $X$ is characterized by a membership function $\mu_A:X\rightarrow[0,1]$ which assigns to each element of $X$ a grade of membership to the set $A$. The support and the core of $A$ are defined, respectively, as the crisp sets $supp(A)=\{x\in X;\ \mu_A(x)>0\}$ and $core(A)=\{x\in X;\ \mu_A(x)=1\}$. A fuzzy set $A$ is normal if its core is nonempty, that is if $height\ A=max_{x\in X}\,\mu_A(x)=1$.

A fuzzy number $A$ is a fuzzy set of the real line with a normal, convex and upper-semicontinuous membership function of bounded support. The $\alpha$-cut of $A$, $0<\alpha\leq 1$, is defined as the crisp set $A_\alpha=\{x\in X;\ \mu_A(x)\geq\alpha\}$ if $0<\alpha\leq 1$ and as the closure of the support if $\alpha=0$. Every $\alpha$-cut of a fuzzy number is a closed interval $A_\alpha=[a_L(\alpha),a_R(\alpha)]$ for $0\leq\alpha\leq 1$, where $a_L(\alpha)=inf\,A_\alpha$ and $a_R(\alpha)=sup\,A_\alpha$.

In the following we deal with *generalized fuzzy numbers*, that is fuzzy numbers whose height may be different from 1. Thus a generalized fuzzy number may be non-normal.

Following the idea expressed in [5] we consider fuzzy quantities obtained by the union of two or more fuzzy sets with non-disjoint supports.

**Definition 1.** We call fuzzy quantity the union of two, or more, generalized fuzzy numbers with non-disjoint supports.

A fuzzy quantity is usually non-convex and non-normal since the union operation on fuzzy sets may produce fuzzy quantities that are non-convex.

In [5] the authors introduced an evaluation for fuzzy quantities based on $\alpha$-cut levels and depending on two parameters: a real number connected with the optimistic/pessimistic point of view of the decision maker and an additive measure that allows the decision maker to attribute different weights to each level, according to his preference. Their definition can be utilized for defuzzification since the evaluation lies in the support of the fuzzy quantity and this is a fundamental property from the point of view of defuzzification problem. In the following we review some of those results. We refer to [5] for more details.

Let $S$ be an additive measure on $[0,1]$ reflecting the subjective attribution of weights to each level $\alpha$ by decision maker and $\lambda\in[0,1]$ be an optimistic/pessimistic parameter.

We assume that $S$ is a normalized Stieltjes measure on $[0,1]$ defined through the function $s$, i.e. $S(]a,b])=s(b)-s(a)$, $0\leq a<b\leq 1$, where $s:[0,1]\rightarrow[0,1]$ is a strictly increasing and continuous function such that $s(0)=0$ and $s(1)=1$.

**Definition 2.** If $A$ is a generalized fuzzy number with $height\ A=w_A\leq 1$ and α-cuts $A_\alpha=[a_L(\alpha),a_R(\alpha)]$, $\alpha\in[0, w_A]$, we define the lower and upper values of $A$

$$V_*(A;S)=\frac{1}{s(w_A)}\int_0^{w_A} a_L(\alpha)dS(\alpha),$$

$$V^*(A;S)=\frac{1}{s(w_A)}\int_0^{w_A} a_R(\alpha)dS(\alpha)$$

and the value of $A$ [5]

$$V_\lambda(A;S)=\frac{1}{s(w_A)}\int_0^{w_A}\phi_\lambda(A_\alpha)dS(\alpha)$$
$$=(1-\lambda)V_*(A;S)+\lambda V^*(A;S)$$

where $\phi_\lambda([x_1,x_2])=(1-\lambda)\ x_1+\lambda x_2$, $x_1\leq x_2$, is an evaluation function.

Note that $V_*(A;S)$, $V^*(A;S)$ and $V_\lambda(A;S)$ belong to the support of $A$. If the measure $S$ is generated by $s(\alpha)=\alpha^r$, $r>0$, we denote $V_\lambda(A;r)= V_\lambda(A;S)$.

**Proposition 1.** Let $A,B$ be two generalized fuzzy numbers with the same height $w_A=w_B$ and let $k$ be a real number. Then, for $\lambda\in[0,1]$

(i) $V_\lambda(A\oplus B;S)=V_\lambda(A;S)+V_\lambda(B;S)$

(ii) $V_\lambda(kA;S)=\begin{cases} k\,V_\lambda(A;S) & k>0 \\ k\,V_{1-\lambda}(A;S) & k<0\end{cases}$

where ⊕ is defined by Zadeh's extension principle.

**Definition 3.** Let $B,C$ be two generalized fuzzy numbers with height $w_B$ and $w_C$, respectively, such that $\overline{\operatorname{supp} B}\cap\overline{\operatorname{supp} C}\neq\emptyset$, where $\overline{\operatorname{supp} B}$ is the closure of the support of $B$. The value of the fuzzy quantity $A=B\cup C$ is

$$V_\lambda(A;S)=V_\lambda(B\cup C;S)$$
$$=\sigma_1 V_\lambda(B;S)+\sigma_2 V_\lambda(C;S)-\sigma_3 V_\lambda(B\cap C;S)$$

where $\sigma_i=\sigma_i(w_B,\ w_C,w_{B\cap C})=\psi_i(s(w_B),\ s(w_C),\ s(w_{B\cap C}))$, $i=1,2,3$, with $\psi_i(z_1,z_2,z_3)=z_i/(z_1+z_2-z_3)$, $z_1+z_2-z_3\neq 0$.
Note that $\sigma_i\geq 0$ and $\sigma_1+\sigma_2-\sigma_3=1$.

**Remark 1.** If $B,C$ are two generalized fuzzy numbers with height $w_B$ and $w_C$, respectively, then $B\cap C$ is a generalized fuzzy number with height $w_{B\cap C}$. Thus the previous definition is well-posed.

Furthermore, $V_\lambda(A;S)$ belongs to the support of the fuzzy quantity $A$.

### 3. Ambiguity of a fuzzy quantity

**Definition 4.** We define the ambiguity of a generalized fuzzy number $A$ with respect to $S$ as

$$Amb(A;S)=\frac{1}{s(w_A)}\int_0^{w_A}\frac{a_R(\alpha)-a_L(\alpha)}{2}dS(\alpha).$$

If the measure $S$ is generated by $s(\alpha)=\alpha^r$, $r>0$, we denote $Amb(A;r)=Amb(A;S)$.

**Proposition 2.** If $A,B$ are generalized fuzzy numbers

$$A\subset B\Rightarrow Amb(A;S)\leq Amb(B;S).$$

**Proposition 3.** If $A$ *is* a generalized fuzzy number

$$Amb(A;S)=\frac{V^*(A;S)-V_*(A;S)}{2}s(w_A).$$

**Proposition 4.** Let $A,B$ be two generalized fuzzy numbers with the same height $w_A=w_B$ and let $k$ be a real number. Then, for $\lambda\in[0,1]$

(i) $Amb(A\oplus B;S)=Amb(A;S)+Amb(B;S)$,

(ii) $Amb(kA;S)=|k|Amb(A;S)$.

**Proposition 5.** Let $A^S$ be the generalized fuzzy number with membership function $\mu_{A^S}(x)=s(\mu_A(x))$, $x\in X$. Then we have $[V_*(A;S),\ V^*(A;S)]=[V_*(A^S;1),\ V^*(A^S;1)]$ and $V_\lambda(A;S)= V_\lambda(A^S;1)$, $Amb(A;S)=Amb(A^S;1)$.

If $s(\alpha)=\alpha^r$ we denote by $A^r=A^S$ the generalized fuzzy number defined by $\mu_{A^r}(x)=(\mu_A(x))^r$.

**Remark 2.** In [8] the operator of concentration/dilation $F_r(A)$, $r>0$, for a fuzzy set $A$ is defined by the membership function $\mu_{F_r(A)}(x)=(\mu_A(x))^r$. If $r=1$ then $F_1(A)=A$. If $r>1$ the modified fuzzy set is a concentration of $A$, that is the reduction in the magnitude of the grade of membership is small for those elements which have a high grade of membership in $A$ and large for the elements with low membership. If $0<r<1$ the modified fuzzy set is a dilation of $A$. The effect of dilation is the opposite of that of concentration. Concentration by $r=2$ is interpreted as the linguistic hedge *very* and dilation by $r=0.5$ *more or less*.

**Definition 5.** We call ambiguity of the fuzzy quantity $A$ the real number

(1) $$Amb(A;S)=\frac{1}{2}\int_0^{w_A} m(A_\alpha)dS(\alpha)$$

where $w_A$ is the height of $A$ and $m(.)$ is the Lebesgue measure on the real line.
If $s(\alpha)=\alpha^r$, we denote $Amb(A;r)= Amb(A;S)$.

**Proposition 6.** If $A,B$ are fuzzy quantities then

$$A\subset B\Rightarrow Amb(A;S)\leq Amb(B;S).$$

*Proof:* Since $A\subset B$ we have $A_\alpha\subset B_\alpha$ and $w_A\leq w_B$. Then $m(A_\alpha)\leq m(B_\alpha)$ and

$$\int_0^{w_A} m(A_\alpha)dS(\alpha)\leq\int_0^{w_A} m(B_\alpha)dS(\alpha)\leq\int_0^{w_B} m(B_\alpha)dS(\alpha)$$

from which it follows that $Amb(A;S)\leq Amb(B;S)$. ■

**Proposition 7.** The ambiguity of the fuzzy quantity $A=B\cup C$ can be expressed as

$$Amb(B\cup C;S)=Amb(B;S)+Amb(C;S)$$
$$-Amb(B\cap C;S)$$

*Proof:* Taking into account that $(B\cup C)_\alpha=B_\alpha\cup C_\alpha$ we obtain $m(A_\alpha)=m(B_\alpha)+m(C_\alpha)-m(B_\alpha\cap C_\alpha)$ and the claim follows from (1) by using the linearity of the Riemann-Stieltjes integral. ■

We call evaluation interval of the fuzzy quantity $A=B\cup C$ the interval $[V_*(A;S),\ V^*(A;S)]$ where the lower and upper values are defined, respectively, by

$$V_*(A;S)=V_*(B\cup C;S)$$
$$=\sigma_1V_*(B;S)+\sigma_2V_*(C;S)-\sigma_3V_*(B\cap C;S),$$
$$V^*(A;S)=V^*(B\cup C;S)$$
$$=\sigma_1V^*(B;S)+\sigma_2V^*(C;S)-\sigma_3V^*(B\cap C;S).$$

Note that $V_\lambda(A;S)=(1-\lambda)V_*(A;S)+\lambda V^*(A;S)$.

**Proposition 8.**

$$Amb(B\cup C;S)=\frac{V^*(B\cup C;S)-V_*(B\cup C;S)}{2}h$$

where $h=h(w_B,w_C,w_{B\cap C})=s(w_B)+s(w_C)-s(w_{B\cap C})$.

**Remark 3.** Since $(B\cup C)^r=B^r\cup C^r$ and $(B\cap C)^r=B^r\cap C^r$ it is easily seen that the value and ambiguity of a fuzzy quantity $A=B\cup C$ with respect to the measure $S$ generated by $s(\alpha)=\alpha^r$ are the same of those of the fuzzy quantity $F_r(A)=(B\cup C)^r=B^r\cup C^r$ with respect to the Lebesgue measure. Thus we have

(i) $V_\lambda(B\cup C;r)=V_\lambda((B\cup C)^r;1)$,

(ii) $Amb(B\cup C;r)=Amb((B\cup C)^r;1)$.

**Example 1.** If

$$A=(a_1,a_2,a_3,a_4,a_5,a_6,a_7;w_1,w_2,w_3),$$

with $w_2<min\{w_1,w_3\}$, is the fuzzy quantity defined by the membership function (see Fig.1)

$$(2)\qquad \mu_A(X)=\begin{cases}\dfrac{w_1}{a_2-a_1}(x-a_1) & a_1\le x\le a_2\\ w_1 & a_2\le x\le a_3\\ \dfrac{w_1-w_2}{a_4-a_3}(a_4-x)+w_2 & a_3\le x\le a_4\\ \dfrac{w_3-w_2}{a_5-a_4}(x-a_4)+w_2 & a_4\le x\le a_5\\ w_3 & a_5\le x\le a_6\\ \dfrac{w_3}{a_7-a_6}(a_7-x) & a_6\le x\le a_7\\ 0 & otherwise\end{cases}$$

then

$$V_*(A;r)=\gamma_1\left[a_1+\frac{r}{r+1}(a_2-a_1)\right]-\gamma_2a_4+\gamma_3\left[a_4+\frac{r}{r+1}g(w_3;w_2,r)(a_5-a_4)\right]$$

$$V^*(A;r)=\gamma_1\left[a_4-\frac{r}{r+1}g(w_1;w_2,r)(a_4-a_3)\right]-\gamma_2a_4+\gamma_3\left[a_7-\frac{r}{r+1}(a_7-a_6)\right]$$

where $\gamma_i=w_i^r/h(w_1,w_2,w_3;r)$, $i=1,2,3$, $h(w_1,w_2,w_3;r)=w_1^r+w_3^r-w_2^r$, and

$$g(w;w_2,r)=1-\frac{w_2}{r(w-w_2)}\left[1-\left(\frac{w_2}{w}\right)^r\right],\ w=w_1,w_3.$$

The value and ambiguity of $A$ can be computed, respectively, by

$$V_\lambda(A;r)=(1-\lambda)V_*(A;r)+\lambda V^*(A;r),$$

$$Amb(A;r)=\frac{V^*(A;r)-V_*(A;r)}{2}h(w_1,w_3,w_2;r).$$

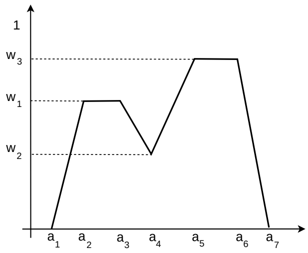


Fig.1. Fuzzy quantity

## 4. Ranking fuzzy quantities

In order to compare two or more fuzzy quantities we introduce a function that maps the set of fuzzy quantities into $\Re^2$ by assigning to every fuzzy quantity $A$ the pair $(V(A),Amb(A))$ where $V(A)$ and $Amb(A)$ are, respectively, the value and ambiguity of $A$ with respect to parameters $\lambda,S$ (fixed).

We use the ambiguity as the degree of ordering in the case that the values of the two fuzzy quantities are equal. Since ambiguity is a measure of the vagueness, that is the lack of precision in determining the exact value of a magnitude, a fuzzy quantity is smaller as its ambiguity is greater. The ranking method we propose can be summarized into the following steps:

1. For two fuzzy quantities $A$ and $B$: if $V(A)>V(B)$ then A>B; if $V(A)<V(B)$ then $A<B$; if $V(A)=V(B)$ then go to the next step.
2. If $Amb(A)<Amb(B)$ then $A>B$ ; if $Amb(A)>Amb(B)$ then $A<B$; if $Amb(A)=Amb(B)$ then $A\sim B$, that is $A$ and $B$ are indifferent.

The proposed ranking method satisfies axioms A1–A5 proposed in [9] as reasonable properties for the rationality of a ranking method for the ordering of fuzzy quantities.

As an application, we use eight sets of fuzzy quantities to illustrate the working of the proposed ranking method. The eight sets of fuzzy quantities are shown in Figg. 2-9. We assume an optimism/pessimism coefficient $\lambda=0.5$. The results of ranking are shown in Table 1 and Table 2 for $s(\alpha)=\alpha$ and $s(\alpha)=\alpha^2$ , respectively. Note that the fuzzy quantities shown in Set 7 and in Set 8 are of the type (2).

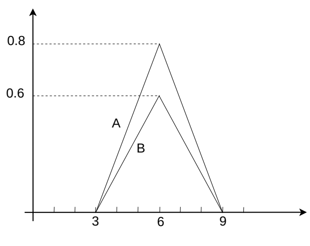


Fig.2. *Set 1:* $A=(3,6,9;0.8)$, $B=(3,6,9;0.6)$

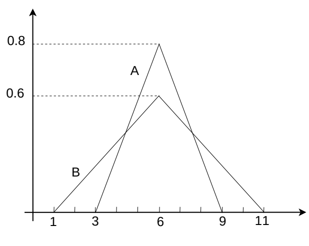


Fig.3. *Set 2:* $A=(3,6,9;0.8)$, $B=(1,6,11;0.6)$

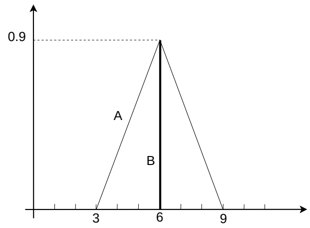


Fig.4. *Set 3:* $A=(3,6,9;0.9),\ B=(6,6,6;0.9)$

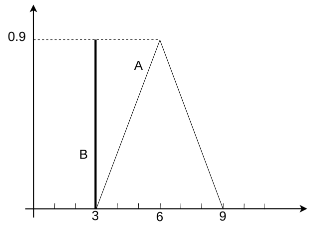


Fig.5. *Set 4:* $A=(3,6,9;0.9),\ B=(3,3,3;0.9)$

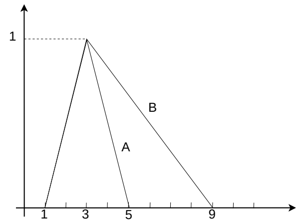


Fig.6. *Set 5:* $A=(1,3,5;1),\ B=(1,3,9;1)$

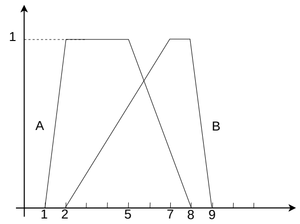


Fig.7. *Set 6:* $A=(1,2,5,8;1),\ B=(2,7,8,9;1)$

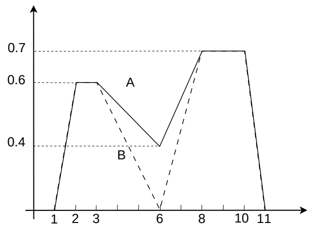


Fig.8. Set 7:
$A=(1,2,3,6,8,10,11;0.6,0.4,0.7),\ B=(1,2,3,6,8,10,11;0.6,0,0.7)$

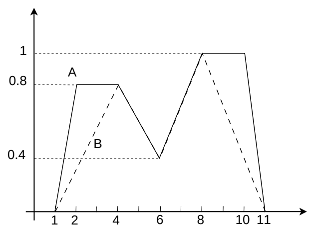


Fig.9. *Set 8:*
$A=(1,2,4,6,8,10,11;0.8,0.4,1),\ B=(1,4,4,6,8,8,11;0.8,0.4,1)$

Table 1. Results of ranking for $\lambda=0.5$ and $r=1$

| Sets | $V(A)$ | $Amb(A)$ | $V(B)$ | $Amb(B)$ | Results |
|---|---|---|---|---|---|
| Set 1 | 6.00 | 1.20 | 6.00 | 0.90 | $B>A$ |
| Set 2 | 6.00 | 1.20 | 6.00 | 1.50 | $A>B$ |
| Set 3 | 6.00 | 1.35 | 6.00 | 0.00 | $B>A$ |
| Set 4 | 6.00 | 1.35 | 3.00 | 0.00 | $A>B$ |
| Set 5 | 3.00 | 1.00 | 4.00 | 2.00 | $B>A$ |
| Set 6 | 4.00 | 2.50 | 6.50 | 2.00 | $B>A$ |
| Set 7 | 6.25 | 2.63 | 6.10 | 2.13 | $A>B$ |
| Set 8 | 6.39 | 3.55 | 6.32 | 2.65 | $A>B$ |

Table 2. Results of ranking for $\lambda=0.5$ and $r=2$

| Sets | $V(A)$ | $Amb(A)$ | $V(B)$ | $Amb(B)$ | Results |
|---|---|---|---|---|---|
| Set 1 | 6.00 | 0.64 | 6.00 | 0.36 | $B>A$ |
| Set 2 | 6.00 | 0.64 | 6.00 | 0.60 | $B>A$ |
| Set 3 | 6.00 | 0.81 | 6.00 | 0.00 | $B>A$ |
| Set 4 | 6.00 | 0.81 | 3.00 | 0.00 | $A>B$ |
| Set 5 | 3.00 | 0.67 | 3.67 | 1.33 | $B>A$ |
| Set 6 | 3.83 | 2.17 | 6.83 | 1.50 | $B>A$ |
| Set 7 | 6.44 | 1.50 | 6.29 | 1.16 | $A>B$ |
| Set 8 | 6.67 | 2.81 | 6.51 | 1.71 | $A>B$ |

## 5. Conclusion

In this article we studied the problem of evaluating and ranking fuzzy quantities, where a fuzzy quantity is usually a non-normal and non-convex fuzzy set, defined as the union of two, or more, generalized fuzzy numbers. To this aim we introduced a definition of ambiguity of non-normal and non-convex fuzzy membership functions. Relations between value and ambiguity were also investigated.

In our view, this framework can be also employed to other types of fuzzy sets characterized by complex shaped membership functions. For instance, our procedure can be used for the evaluation and ranking of non-convex and non-normal intuitionistic fuzzy sets. This will be a topic of our future research work.

*Wydanie publikacji zrealizowano przy udziale środków finansowych otrzymanych z budżetu Województwa Zachodniopomorskiego.*

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