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Positive stable descriptor continuous-discrete 2D linear system

***Abstract**. The positivity and asymptotic stability of descriptor continuous-discrete 2D linear systems are addressed. Two different groups of sufficient conditions for the positivity and asymptotic stability of the system are established. Two procedures for checking positivity and asymptotic stability of the systems are proposed and illustrated by numerical examples.*

***Streszczenie.** W pracy rozpatrywana jest dodatniość i stabilność asymptotyczna singularnych (deskryptorowych) dwuwymiarowych (2D) układów ciągło-dyskretnych. Podano dwie grupy warunków wystarczających dodatniości i asymptotycznej stabilności dla tej klasy układów. Podano też dwie procedury pozwalające sprawdzić dodatniość i asymptotyczną stabilność tych układów. Proponowane procedury zostały zilustrowane przykładami numerycznymi. (**Dodatnie stabilne singularne ciągło-dyskretne liniowe układy 2D**).*

Keywords: Positive, descriptor, continuous-discrete, stability.
Słowa kluczowe: dodatni, singularny, ciągło-dyskretny, stabilność.

Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. A variety of models having positive systems behavior can be found in engineering, management science, economics, social sciences, biology and medicine etc.. An overview of state of the art in positive systems is given in the monographs [5, 8]. The positive continuous-discrete 2D linear systems have been introduced in [7], positive hybrid linear systems in [9] and the positive fractional 2D hybrid systems in [10]. Different methods of solvability of 2D hybrid linear systems have been discussed in [15] and the solution to singular 2D hybrids linear systems has been derived in [17]. The realization problem for positive 2D hybrid systems has been addressed in [11]. Some problems of dynamics and control of 2D hybrid systems have been considered in [4, 6]. The problems of stability and robust stability of 2D continuous-discrete linear systems have been investigated in [1-3, 18-20]. Recently the stability and robust stability of general model and of Roesser type model of scalar continuous-discrete linear systems have been analyzed by Busłowicz in [2, 3]. Stability of continuous-discrete 2D linear systems has been considered in [13] and of descriptor positive linear systems in [14].

In this paper new sufficient conditions for the positivity and asymptotic stability of descriptor continuous-discrete 2D linear systems will be presented. The paper is organized as follows. In section 2 some preliminaries concerning the continuous-discrete 2D linear systems are recalled. Two different groups of sufficient conditions for the positivity and asymptotic stability of the descriptor systems are presented in section 3. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ - the set of real numbers, Z_+ - the set of nonnegative integers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, I_n - the $n\times n$ identity matrix.

Preliminaries

Consider the linear continuous-discrete 2D system [8, 13]

(1) $\dot{x}(t,i+1) = A_0 x(t,i) + A_1\dot{x}(t,i) + A_2 x(t,i+1) + Bu(t,i)$

where $t\in\Re_+$, $i\in Z_+$ and $\dot{x}(t,i) = \dfrac{\partial x(t,i)}{\partial t}$, $x(t,i)\in\Re^n$, $u(t,i)\in\Re^m$ $A_0, A_1, A_2 \in \Re^{n\times n}$, $B\in\Re^{n\times m}$.

Definition 1. The continuous-discrete 2D linear system (1) is called (internally) positive if $x(t,i)\in\Re_+^n$, $t\in\Re_+$, $i\in Z_+$ for any input $u(t,i)\in\Re_+^m$ and all initial conditions

(2) $x(0,i)\in\Re_+^n,\ i\in Z_+,\ x(t,0)\in\Re_+^n,\ \dot{x}(t,0)\in\Re_+^n,\ t\in\Re_+$

Theorem 1. [7, 8] The continuous-discrete 2D linear system (1) is positive if and only if

(3) $A_2\in M_n,\ A_0, A_1\in\Re_+^{n\times n},\ A_0 + A_1A_2\in\Re_+^{n\times n},\ B\in\Re_+^{n\times m}$

where M_n is the set of $n\times n$ Metzler matrices (with nonnegative off-diagonal entries).

Definition 2. The continuous-discrete 2D linear system (1) is called asymptotically stable if

(4) $$\lim_{t,i\to\infty} x(t,i) = 0 .$$

A matrix $A\in\Re^{n\times n}$ is called asymptotically stable (Hurwitz) if all its eigenvalues lie in the open left half of the complex plane.

Theorem 2. [13] The positive continuous-discrete 2D linear system (1) is asymptotically stable if and only if all coefficients of the polynomial

(5) $$\begin{aligned}&\det[I_n s(z+1) - A_0 - A_1 s - A_2(z+1)]\\ &= s^n z^n + \bar{a}_{n,n-1}s^n z^{n-1} + \bar{a}_{n-1,n}s^{n-1}z^n + \ldots + \bar{a}_{10}s + \bar{a}_{01}z + \bar{a}_{00}\end{aligned}$$

are positive, i.e.

(6) $$\bar{a}_{k,l} > 0 \text{ for } k, l = 0,1,\ldots,n \ (\bar{a}_{n,n} = 1).$$

Theorem 3. The positive continuous-discrete 2D linear system (1) is asymptotically stable if and only if both matrices

(7) $$A_1 - I_n,\ A_0 + A_2$$

are Hurwitz Metzler matrix.

Proof is given in [13].

Theorem 4. [5, 8, 12] The matrix $A\in\Re^{n\times n}$ is Hurwitz Metzler matrix if and only if all coefficients of the characteristic polynomial

(8) $\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0$

are positive, i.e. $a_k \geq 0$ for $k = 0,1,\ldots,n\text{-}1$.

Descriptor continuous-discrete 2D linear systems

Consider the descriptor continuous-discrete 2D linear system

(9) $E\dot{x}(t,i+1) = A_0 x(t,i) + A_1\dot{x}(t,i) + A_2 x(t,i+1) + Bu(t,i)$

where $t \in \Re_+$, $i \in Z_+$ and $\dot{x}(t,i) = \frac{\partial x(t,i)}{\partial t}$, $x(t,i) \in \Re^n$, $u(t,i) \in \Re^m$ $E, A_0, A_1, A_2 \in \Re^{n\times n}$, $B \in \Re^{n\times m}$. It is assumed that $\det E = 0$ and

(10) $p(s) = \det[Es - A_0 - A_1 s - A_2 z] \neq 0$ for some $s \in \mathrm{C}$

where C is the field of complex numbers.

Definition 3. The descriptor continuous-discrete 2D linear system (9) is called (internally) positive if for every consistent nonnegative conditions $x(0,i) \in \Re_+^n$, $i \in Z_+$, $x(t,0) \in \Re_+^n$, $\dot{x}(t,0) \in \Re_+^n$ and all inputs $u(t,i) \in \Re_+^m$, $t \in \Re_+$, $i \in Z_+$, $x(t,i) \in \Re_+^n$ for $t \in \Re_+$, $i \in Z_+$.

The following elementary row (column) operations will be used [8, 14]:

– Multiplication of the *i*th row (column) by a real number c. This operation will be denoted by $L[i \times c]$ ($R[i \times c]$).
– Addition to the *i*th row (column) of the *j*th row (column) multiplied by a real number c. This operation will be denoted by $L[i + j \times c]$ ($R[i + j \times c]$).
– Interchange of the *i*th and *j*th rows (columns). This operation will be denoted by $L[i, j]$ ($R[i, j]$).

It is assumed that using elementary row and column operations it is possible to reduce the matrices $E, A_0, A_1, A_2 \in \Re^{n\times n}$ to the form

(11) $P[Esz - A_0 - A_1 s - A_2 z]Q = \overline{E}sz - \overline{A}_0 - \overline{A}_1 s - \overline{A}_2 z$

where

$$\overline{E} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix},\ \overline{A}_k = \begin{bmatrix} A_{11}^k & A_{12}^k \\ A_{21}^k & A_{22}^k \end{bmatrix},\ k = 0,1,2$$

(12) $A_{11}^k \in \Re^{n_1\times n_1},\ A_{12}^k \in \Re^{n_1\times n_2},\ A_{21}^k \in \Re^{n_2\times n_1},\ A_{22}^k \in \Re^{n_2\times n_2},$
$n_1 = \mathrm{rank}E < n,\ n_2 = n - n_1$

and P is a matrix of elementary row operations and $Q \in \Re^{n\times n}$ is a monomial matrix of elementary column operations. The matrix P can be obtained by performing the elementary row operations and the matrix Q by performing the elementary column operations on the identity matrix I_n, respectively [8, 14].

Note that if Q is a monomial matrix then

(13) $\overline{x}(t,i) = Q^{-1}x(t,i) \in \Re_+^n$ for $t \in \Re_+$, $i \in Z_+$

for $x(t,i) \in \Re_+^n$, $t \in \Re_+$, $i \in Z_+$ since $Q^{-1} \in \Re_+^{n\times n}$.

Premultiplying (9) by the matrix P and defining the new state vector

(14) $\overline{x}(t,i) = Q^{-1}x(t,i) = \begin{bmatrix} x_1(t,i) \\ x_2(t,i) \end{bmatrix},\ x_1(t,i) \in \Re_+^{n_1},\ x_2(t,i) \in \Re_+^{n_2}$

we obtain

(15)
$$\begin{aligned} PEQQ^{-1}\dot{x}(t,i+1) &= \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1(t,i+1) \\ \dot{x}_2(t,i+1) \end{bmatrix} \\ &= PA_0QQ^{-1}x(t,i) + PA_1QQ^{-1}\dot{x}(t,i) + PA_2QQ^{-1}x(t,i+1) \\ &+ PBu(t,i) = \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix}\begin{bmatrix} x_1(t,i) \\ x_2(t,i) \end{bmatrix} + \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix}\begin{bmatrix} \dot{x}_1(t,i) \\ \dot{x}_2(t,i) \end{bmatrix} \\ &+ \begin{bmatrix} A_{11}^2 & A_{12}^2 \\ A_{21}^2 & A_{22}^2 \end{bmatrix}\begin{bmatrix} x_1(t,i+1) \\ x_2(t,i+1) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u(t,i) \end{aligned}$$

and

(16)
$$\begin{aligned} \dot{x}_1(t,i+1) &= A_{11}^0 x_1(t,i) + A_{12}^0 x_2(t,i) + A_{11}^1\dot{x}_1(t,i) + A_{12}^1\dot{x}_2(t,i) \\ &+ A_{11}^2 x_1(t,i+1) + A_{12}^2 x_2(t,i+1) + B_1 u(t,i) \end{aligned}$$

(17)
$$\begin{aligned} 0 &= A_{21}^0 x_1(t,i) + A_{22}^0 x_2(t,i) + A_{21}^1\dot{x}_1(t,i) + A_{22}^1\dot{x}_2(t,i) \\ &+ A_{21}^2 x_1(t,i+1) + A_{22}^2 x_2(t,i+1) + B_2 u(t,i) \end{aligned}$$

where

(18) $PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ B_1 \in \Re^{n_1\times m},\ B_2 \in \Re^{n_2\times m}$.

We shall consider the following two cases.

Case 1.

The matrix A_{22}^0 is a stable Metzler matrix. In this case $-(A_{22}^0)^{-1} \in \Re_+^{n_2\times n_2}$ and from (17) we obtain

(19)
$$\begin{aligned} x_2(t,i) &= -(A_{22}^0)^{-1}[A_{21}^0 x_1(t,i) + A_{21}^1\dot{x}_1(t,i) + A_{22}^1\dot{x}_2(t,i) \\ &+ A_{21}^2 x_1(t,i+1) + A_{22}^2 x_2(t,i+1) + B_2 u(t,i)]. \end{aligned}$$

Substitution of (19) into (16) yields

(20) $\dot{x}_1(t,i+1) = A_1^0 x_1(t,i) + A_1^1\dot{x}_1(t,i) + A_1^2 x_1(t,i+1) + B_1^0 u(t,i)$

where

(21)
$$\begin{aligned} &A_1^0 = A_{11}^0 - A_{12}^0(A_{22}^0)^{-1}A_{21}^0,\ A_1^1 = A_{11}^1 - A_{12}^0(A_{22}^0)^{-1}A_{21}^1, \\ &A_1^2 = A_{11}^2 - A_{12}^0(A_{22}^0)^{-1}A_{21}^2,\ B_1^0 = B_1 - A_{12}^0(A_{22}^0)^{-1}B_2 \end{aligned}$$

and under the assumption

(22) $A_{12}^1 - A_{12}^0(A_{22}^0)^{-1}A_{22}^1 = 0,\ A_{12}^2 - A_{12}^0(A_{22}^0)^{-1}A_{22}^2 = 0$.

Assuming

(23) $A_{21}^1 = 0,\ A_{22}^1 = 0,\ A_{22}^2 = 0$

from (19) we obtain

(24) $x_2(t,i) = A_2^0 x_1(t,i) + A_2^2 x_1(t,i+1) + B_2^0 u(t,i)$

where

(25) $A_2^0 = -(A_{22}^0)^{-1} A_{21}^0,\ A_2^2 = -(A_{22}^0)^{-1} A_{21}^2,\ B_2^0 = -(A_{22}^0)^{-1} B_2$.

By Theorem 1 the system (20) is positive if and only if

$$A_1^2 \in M_{n_1},\ A_1^0, A_1^1 \in \Re_+^{n_1 \times n_1}, \quad A_1^0 + A_1^1 A_1^2 \in \Re_+^{n_1 \times n_1},\ B_1^0 \in \Re_+^{n_1 \times m}. \tag{26}$$

If $x_1(t,i) \in \Re_+^{n_1}$ for $t \in \Re_+$, $i \in Z_+$ then from (24) it follows that $x_2(t,i) \in \Re_+^{n_2}$ for $t \in \Re_+$, $i \in Z_+$ and

$$A_2^0, A_2^2 \in \Re_+^{n_2 \times n_2} \text{ and } B_2^0 \in \Re_+^{n_2 \times m} \tag{27}$$

since $u(t,i) \in \Re_+^m$ for $t \in \Re_+$ and $i \in Z_+$.

Theorem 5. Let there exist a pair of matrices of elementary operations $P \in \Re^{n \times n}$ and monomial elementary column operations matrix $Q \in \Re^{n \times n}$ satisfying (14) and (15). Then the descriptor continuous-discrete system (9) is positive and asymptotically stable if $A_{22}^0 \in M_{n_2}$ is asymptotically stable and

$$\begin{gathered} A_{11}^0 \in \Re_+^{n_1 \times n_1},\ A_{12}^0 \in \Re_+^{n_1 \times n_2},\ A_{21}^0 \in \Re_+^{n_2 \times n_1},\ A_{11}^1 \in \Re_+^{n_1 \times n_1}, \\ A_{21}^1 \in \Re_+^{n_2 \times n_1},\ A_{11}^2 \in M_{n_1},\ A_{21}^2 \in \Re_+^{n_2 \times n_1}, \\ B_1 \in \Re_+^{n_1 \times m},\ B_2 \in \Re_+^{n_2 \times m} \end{gathered} \tag{28}$$

the conditions (26) are satisfied and the matrices $A_1^1 - I_{n_1}$, $A_1^0 + A_1^2$ are asymptotically stable.

Proof. From (21) it follows that $A_1^0, A_1^1 \in \Re_+^{n_1 \times n_1}$, $B_1^0 \in \Re_+^{n_1 \times m}$ and $A_1^2 \in M_{n_1}$ if (28) holds since by assumption $A_{22}^0 \in M_{n_2}$ is asymptotically stable and $-(A_{22}^0)^{-1} \in \Re_+^{n_2 \times n_2}$ [9]. Thus, by Theorem 1 the systems (20) is positive and $x_1(t,i) \in \Re_+^{n_1}$ for $t \in \Re_+$, $i \in Z_+$. If (28) holds then from (25) we have $A_2^0, A_2^2 \in \Re_+^{n_2 \times n_2}$ and $B_2^0 \in \Re_+^{n_2 \times m}$. In this case from (24) it follows that $x_2(t,i) \in \Re_+^{n_2}$ for $x_1(t,i) \in \Re_+^{n_1}$ and $u(t,i) \in \Re_+^m$ for $t \in \Re_+$, $i \in Z_+$. By Theorem 3 the system (20) is asymptotically stable if and only if the matrices $A_1^1 - I_{n_1}$, $A_1^0 + A_1^2$ are asymptotically stable (are Hurwitz Metzler Matrices). If $\lim_{t,i \to \infty} x_1(t,i) = 0$ then from (24) for $u(t,i) = 0$ we have also $\lim_{t,i \to \infty} x_2(t,i) = 0$. This completes the proof. □

If the conditions of Theorem 5 are satisfied then the following procedure can be used to check the positivity and asymptotic stability of the systems (9).

Procedure 1.

Step 1. Using the elementary row and column operations reduce the matrices of the system (9) to the form (15) and find the matrices P and Q.

Step 2. Knowing $A_{i,j}^k$ for $i, j = 1,2$ and $k = 0,1,2$ compute the matrices (21) and (25). If the conditions (28) and (26) are met then the system (9) is positive.

Step 3. Compute the matrices

$$\hat{A}_1 = A_1^1 - I_{n_1},\ \hat{A}_2 = A_1^0 + A_1^2. \tag{29}$$

If the matrices (29) are Hurwitz Metzler matrices then the positive system (9) is also asymptotically stable.

Example 1. Consider the system (9) with the matrices

$$\begin{gathered} E = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix},\ A_0 = \begin{bmatrix} 1 & 0.4375 & 0.1 & 2 \\ 0 & 0.525 & 1 & 1 \\ -4 & 0.125 & 0.8 & -3 \\ 0 & 1.1 & 2 & 0 \end{bmatrix}, \\ A_1 = \begin{bmatrix} 0 & 0.25 & 0.1 & 0 \\ 0 & 0.1 & 0.2 & 0 \\ 0 & -0.5 & -0.2 & 0 \\ 0 & 0.2 & 0.4 & 0 \end{bmatrix}, \\ A_2 = \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \end{gathered} \tag{30}$$

Using Procedure 1 we obtain the following.

Step 1. To reduce the matrices (30) to the desired form (15) we perform the following elementary operations: $L[3+1\times 2]$, $L[4+2\times(-2)]$ and $R[1,2]$, $R[2,3]$, $R[1\times 2]$ and we obtain

$$\begin{gathered} \overline{E} = PEQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \overline{A}_0 = PA_0Q = \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix} = \begin{bmatrix} 0.875 & 0.1 & 1 & 2 \\ 1.05 & 1 & 0 & 1 \\ 2 & 1 & -2 & 1 \\ 0.1 & 0 & 0 & -2 \end{bmatrix}, \\ \overline{A}_1 = PA_1Q = \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 & 0 & 0 \\ 0.2 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ \overline{A}_2 = PA_2Q = \begin{bmatrix} A_{11}^2 & A_{12}^2 \\ A_{21}^2 & A_{22}^2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{gathered} \tag{31}$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = P \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \tag{31}$$

where

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix},\ Q = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{32}$$

Step 2. Using (21) and (25) and taking into account that

$$(A_{22}^0)^{-1}=\begin{bmatrix}-2 & 1\\ 0 & -2\end{bmatrix}^{-1}=\begin{bmatrix}-0.5 & -0.25\\ 0 & -0.5\end{bmatrix}$$

we obtain

$$(33)\quad \begin{aligned}
A_1^0 &= A_{11}^0 - A_{12}^0(A_{22}^0)^{-1}A_{21}^0=\begin{bmatrix}0.875 & 0.1\\ 1.05 & 1\end{bmatrix}\\
&+\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}0.5 & 0.25\\ 0 & 0.5\end{bmatrix}\begin{bmatrix}2 & 1\\ 0.1 & 0\end{bmatrix}=\begin{bmatrix}2 & 0.6\\ 1.1 & 1\end{bmatrix},\\
A_1^1 &= A_{11}^1 - A_{12}^0(A_{22}^0)^{-1}A_{21}^1=\begin{bmatrix}0.5 & 0.1\\ 0.2 & 0.2\end{bmatrix}\\
&+\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}0.5 & 0.25\\ 0 & 0.5\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}0.5 & 0.1\\ 0.2 & 0.2\end{bmatrix},\\
A_1^2 &= A_{11}^2 - A_{12}^0(A_{22}^0)^{-1}A_{21}^2=\begin{bmatrix}-4 & 0\\ 0 & -2\end{bmatrix}\\
&+\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}0.5 & 0.25\\ 0 & 0.5\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}-4 & 0\\ 0 & -2\end{bmatrix},\\
B_1^0 &= B_1 - A_{12}^0(A_{22}^0)^{-1}B_2=\begin{bmatrix}1\\ 0\end{bmatrix}\\
&+\begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\begin{bmatrix}0.5 & 0.25\\ 0 & 0.5\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix}=\begin{bmatrix}3.25\\ 0.5\end{bmatrix}
\end{aligned}$$

and

$$(34)\quad \begin{aligned}
A_2^0 &= -(A_{22}^0)^{-1}A_{21}^0=\begin{bmatrix}0.5 & 0.25\\ 0 & 0.5\end{bmatrix}\begin{bmatrix}2 & 1\\ 0.1 & 0\end{bmatrix}=\begin{bmatrix}1.025 & 0.5\\ 0.05 & 0\end{bmatrix},\\
A_2^2 &= -(A_{22}^0)^{-1}A_{21}^2=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\\
B_2^0 &= -(A_{22}^0)^{-1}B_2=\begin{bmatrix}0.5 & 0.25\\ 0 & 0.5\end{bmatrix}\begin{bmatrix}2\\ 1\end{bmatrix}=\begin{bmatrix}1.25\\ 0.5\end{bmatrix}.
\end{aligned}$$

The conditions (28), (25) and (24) are satisfied since $A_1^2\in M_2$, $A_1^0, A_1^1\in\Re_+^{2\times 2}$ and

$$(35)\quad \begin{aligned}
A_1^0 + A_1^1A_1^2 &= \begin{bmatrix}2 & 0.6\\ 1.1 & 1\end{bmatrix}\\
&+\begin{bmatrix}0.5 & 0.1\\ 0.2 & 0.2\end{bmatrix}\begin{bmatrix}-4 & 0\\ 0 & -2\end{bmatrix}=\begin{bmatrix}0 & 0.4\\ 0.3 & 0.6\end{bmatrix}\in\Re_+^{2\times 2}.
\end{aligned}$$

Thus, the system (9) with (30) is positive.

Step 3. Using (29) we obtain

$$(36)\quad \begin{aligned}
\hat{A}_1 &= A_1^1 - I_{n_1}=\begin{bmatrix}0.5 & 0.1\\ 0.2 & 0.2\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}-0.5 & 0.1\\ 0.2 & -0.8\end{bmatrix}\in M_2,\\
\hat{A}_2 &= A_1^0 + A_1^2=\begin{bmatrix}2 & 0.6\\ 1.1 & 1\end{bmatrix}+\begin{bmatrix}-4 & 0\\ 0 & -2\end{bmatrix}=\begin{bmatrix}-2 & 0.6\\ 1.1 & -1\end{bmatrix}\in M_2.
\end{aligned}$$

The matrices (36) are Hurwitz Metzler matrices and the positive system (9) with (30) is positive and asymptotically stable.

Case 2.

Let $A_{22}^1\in M_{n_2}$ be asymptotically stable then $-(A_{22}^1)^{-1}\in\Re_+^{n_2\times n_2}$ and from (17) we obtain

$$(37)\quad \begin{aligned}
\dot{x}_2(t,i) &= -(A_{22}^1)^{-1}[A_{21}^0x_1(t,i)+A_{22}^0x_2(t,i)+B_2u(t,i)]\\
&= \overline{A}_1^0x_1(t,i)+\overline{A}_2^0x_2(t,i)+\overline{B}_2^0u(t,i)
\end{aligned}$$

for

$$(38)\quad A_{21}^1=0,\ A_{21}^2=0,\ A_{22}^2=0$$

where

$$(39)\quad \overline{A}_1^0=-(A_{22}^1)^{-1}A_{21}^0,\ \overline{A}_2^0=-(A_{22}^1)^{-1}A_{22}^0,\ \overline{B}_2^0=-(A_{22}^1)^{-1}B_2.$$

Substitution of (37) into (17) yields

$$(40)\quad \dot{x}_1(t,i+1)=\widetilde{A}_1^0x_1(t,i)+\widetilde{A}_1^1\dot{x}_1(t,i)+\widetilde{A}_1^2x_1(t,i+1)+\widetilde{B}_1^0u(t,i)$$

where

$$(41)\quad \begin{aligned}
&\widetilde{A}_1^0=A_{11}^0-A_{12}^1(A_{22}^1)^{-1}A_{21}^0,\ \widetilde{A}_1^1=A_{11}^1,\\
&\widetilde{A}_1^2=A_{11}^2,\ \widetilde{B}_1^0=B_1-A_{12}^1(A_{22}^1)^{-1}B_2
\end{aligned}$$

and

$$(42)\quad A_{12}^0-A_{12}^1(A_{22}^1)^{-1}A_{22}^0=0,\ A_{12}^2=0.$$

Assuming that $x_1(t,i)$ and $u(t,i)$ are known then from the equation (37) we may find its solution

$$(43)\quad x_2(t,i)=e^{\overline{A}_2^0t}x_2(0,i)+\int_0^t e^{\overline{A}_2^0(t-\tau)}[\overline{A}_1^0x_1(\tau,i)+\overline{B}_2^0u(\tau,i)]d\tau.$$

By Theorem 1 the system (40) is positive if and only if

$$(44)\quad \begin{aligned}
&\widetilde{A}_1^2\in M_{n_1},\ \widetilde{A}_1^0,\widetilde{A}_1^1\in\Re_+^{n_1\times n_1},\\
&\widetilde{A}_1^0+\widetilde{A}_1^1\widetilde{A}_1^2\in\Re_+^{n_1\times n_1},\ \widetilde{B}_1^0\in\Re_+^{n_1\times m}.
\end{aligned}$$

From (43) it follows that $x_2(t,i)\in\Re_+^{n_2}$ for $t\in\Re_+$, $i\in Z_+$ if the matrix

$$(45)\quad \begin{aligned}
&\overline{A}_2^0\in M_{n_2}\ \text{is asymptotically stable},\\
&\overline{A}_1^0\in\Re_+^{n_2\times n_1}\ \text{and}\ \overline{B}_2^0\in\Re_+^{n_2\times m}.
\end{aligned}$$

Theorem 6. Let there exist a pair of matrices of elementary operations $P\in\Re^{n\times n}$ and monomial elementary column operations matrix $Q\in\Re^{n\times n}$ satisfying (14) and (15). Then the descriptor continuous-discrete system (9) is positive and asymptotically stable if $A_{22}^1\in M_{n_2}$ is asymptotically stable and

$$(46)\quad \begin{aligned}
&A_{11}^0\in\Re_+^{n_1\times n_1},\ A_{21}^0\in\Re_+^{n_2\times n_1},\ A_{22}^0\in\Re_+^{n_2\times n_2},\\
&A_{11}^1\in\Re_+^{n_1\times n_1},\ A_{12}^1\in\Re_+^{n_1\times n_2},\ A_{11}^2\in M_{n_1},\\
&B_1\in\Re_+^{n_1\times m},\ B_2\in\Re_+^{n_2\times m}
\end{aligned}$$

the conditions (44) and (45) are satisfied and the matrices $\widetilde{A}_1^1-I_{n_1}$, $\widetilde{A}_1^0+\widetilde{A}_1^2$ are asymptotically stable.

Proof. From (41) and (42) it follows that $\widetilde{A}_1^0,\widetilde{A}_1^1\in\Re_+^{n_1\times n_1}$, $\widetilde{B}_1^0\in\Re_+^{n_1\times m}$ and $\widetilde{A}_1^2\in M_{n_1}$ if (46) holds since $A_{22}^1\in M_{n_2}$ is asymptotically stable and $-(A_{22}^1)^{-1}\in\Re_+^{n_2\times n_2}$. Thus, by

Theorem 1 the system (40) is positive and $x_1(t,i) \in \Re_+^{n_1}$ for $t \in \Re_+$, $i \in Z_+$. If the condition (45) is met then from (43) we have $x_2(t,i) \in \Re_+^{n_2}$ for $t \in \Re_+$, $i \in Z_+$ since $x_2(0,i) \in \Re_+^{n_2}$ for $x_2(t,i) \in \Re_+^{n_1}$ and $u(t,i) \in \Re_+^m$ for $t \in \Re_+$, $i \in Z_+$. Therefore, the system (9) is positive. By Theorem 3 the system (3.22a) is asymptotically stable if and only if the matrices $\widetilde{A}_1^1 - I_{n_1}$, $\widetilde{A}_1^0 + \widetilde{A}_1^2$ are Hurwitz Metzler Matrices. The system (37) is asymptotically table if (45) holds. Therefore, the system (9) is positive and asymptotically stable. □

If the conditions of Theorem 6 are satisfied then the following procedure can be used to check the positivity and asymptotic stability of the systems (9).

Procedure 2.

Step 1. Using the elementary row and column operations reduce the matrices of the system (9) to the form (15) and find the matrices P and Q.

Step 2. Knowing $A_{i,j}^k$ for $i, j = 1,2$ and $k = 0,1,2$ compute the matrices (41) and (42). If the conditions (44) and (45) are met then the system (9) is positive.

Step 3. Compute the matrices

$$\hat{A}'_1 = \widetilde{A}_1^1 - I_{n_1}, \ \hat{A}'_2 = \widetilde{A}_1^0 + \widetilde{A}_1^2 . \tag{47}$$

If the matrices (47) are Hurwitz Metzler matrices then the positive system (9) is also asymptotically stable.

Example 2. Consider the system (9) with the matrices

$$E = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}, \ A_0 = \begin{bmatrix} 1 & 2.6 & 0.7 \\ -1 & 1.2 & 0.2 \\ 3 & -2.2 & -0.1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -1 & 0.6 & 0.1 \\ 1 & 0.4 & 0 \\ -3 & -0.8 & 0 \end{bmatrix}, \tag{48}$$

$$A_2 = \begin{bmatrix} 0 & -4 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 6 \end{bmatrix}, \ B = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

Using Procedure 2 we obtain the following.

Step 1. To reduce the matrices (48) to the desired form (15) we perform the following elementary operations: $L[3+2\times 2]$, $L[1+3\times(-1)]$, $R[1,3]$, $R[1,2]$, $R[1\times 0.5]$ and we obtain

$$\overline{E} = PEQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\overline{A}_0 = PA_0Q = \begin{bmatrix} A_{11}^0 & A_{12}^0 \\ A_{21}^0 & A_{22}^0 \end{bmatrix} = \begin{bmatrix} 1.2 & 0.4 & 0 \\ 0.6 & 0.2 & -1 \\ 0.1 & 0.3 & 1 \end{bmatrix}, \tag{49}$$

$$\overline{A}_1 = PA_1Q = \begin{bmatrix} A_{11}^1 & A_{12}^1 \\ A_{21}^1 & A_{22}^1 \end{bmatrix} = \begin{bmatrix} 0.3 & 0.1 & 0 \\ 0.2 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix},$$

$$\overline{A}_2 = PA_2Q = \begin{bmatrix} A_{11}^2 & A_{12}^2 \\ A_{21}^2 & A_{22}^2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\overline{B} = PB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

where

$$P = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \ Q = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{50}$$

Step 2. Taking into account that $(A_{22}^1)^{-1} = 1$ and using (41) we obtain

$$\begin{aligned} \widetilde{A}_1^0 &= A_{11}^0 - A_{12}^1 (A_{22}^1)^{-1} A_{21}^0 = \begin{bmatrix} 1.2 & 0.4 \\ 0.6 & 0.2 \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0.1 \quad 0.3] = \begin{bmatrix} 1.2 & 0.4 \\ 0.7 & 0.5 \end{bmatrix}, \\ \widetilde{A}_1^1 &= A_{11}^1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0 \end{bmatrix}, \ \widetilde{A}_1^2 = A_{11}^2 = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \\ \widetilde{B}_1^0 &= B_1 - A_{12}^1 (A_{22}^1)^{-1} B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} 2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned} \tag{51}$$

and

$$\begin{aligned} \overline{A}_1^0 &= -(A_{22}^1)^{-1} A_{21}^0 = [0.1 \quad 0.3], \\ \overline{A}_2^0 &= -(A_{22}^1)^{-1} A_{22}^0 = 1, \\ \overline{B}_2^0 &= -(A_{22}^1)^{-1} B_2 = 2. \end{aligned} \tag{52}$$

The conditions (44) and (45) are satisfied since $\widetilde{A}_1^2 \in M_2$ is asymptotically stable $\widetilde{A}_1^0, \widetilde{A}_1^1 \in \Re_+^{2\times 2}$, $\widetilde{B}_1^0 \in \Re_+^2$, $\widetilde{B}_2^0 \in \Re_+$ and

$$\begin{aligned} \widetilde{A}_1^0 + \widetilde{A}_1^1 \widetilde{A}_1^2 &= \begin{bmatrix} 1.2 & 0.4 \\ 0.7 & 0.5 \end{bmatrix} \\ &+ \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.1 \\ 0.3 & 0.5 \end{bmatrix} \in \Re_+^{2\times 2} \end{aligned} \tag{53}$$

Thus, the system (9) with (48) is positive.

Step 3. Using (47) we obtain

$$\begin{aligned} \hat{A}'_1 &= \widetilde{A}_1^1 - I_{n_1} = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -0.7 & 0.1 \\ 0.2 & -1 \end{bmatrix} \in M_2, \\ \hat{A}'_2 &= \widetilde{A}_1^0 + \widetilde{A}_1^2 = \begin{bmatrix} 1.2 & 0.4 \\ 0.7 & 0.5 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \\ &= \begin{bmatrix} -0.8 & 0.4 \\ 0.7 & -2.5 \end{bmatrix} \in M_2. \end{aligned} \tag{54}$$

The matrices (54) are Hurwitz Metzler matrices and the positive system (9) with (48) is positive and asymptotically stable.

Concluding remarks

Two different groups of sufficient conditions for the positivity and asymptotic stability of the descriptor continuous-discrete 2D linear system have been established. Two procedures for checking positivity and asymptotic stability of the descriptor systems have been proposed. The procedures have been illustrated by numerical examples. The considerations can be extended to fractional descriptor continuous-discrete 2D linear systems. An open problem is necessary and sufficient conditions for the positivity and asymptotic stability of the descriptor continuous-discrete 2D linear systems. An other open problem is an extension of these considerations for the descriptor switched 2D linear systems.

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