Stanisław PAWŁOWSKI[1], Jolanta PLEWAKO[2]

Rzeszow University of Technology, Department of Electrodynamics and Electrical Machine Systems (1), Department of Power Electronics and Power Engineering (2)

Fundamental solutions method applied to non-linear, three-dimensional problems in electromagnetism

Abstract: *The method for solving non-linear 3D magnetostatics problems is presented in the paper. It is based on the iterative version of the fundamental solutions method. A dozen of numerical tests performed on a simple model system provided grounds to evaluate the method as effective and correct*.

Streszczenie:*. W pracy zaprezentowano metodę rozwiązywania nieliniowych 3D zagadnień magnetostatyki opartą na iteracyjnej wersji metody rozwiązań fundamentalnych. Na prostym przykładzie modelowym przeprowadzono szereg testów numerycznych, które wstępnie pozwalają ocenić ją jako poprawną i skuteczną.* (***Zastosowanie metody rozwiązań fundamentalnych w nieliniowych trójwymiarowych zagadnieniach elektromagnetyzmu****).*

Keywords: magnetostatics, non-linear problems, fundamental solutions method, iteration
Słowa kluczowe: magnetostatyka, nieliniowe zagadnienia, metoda fundamentalnych rozwiązań, iteracja

Introduction

Domain methods, such as finite elements method (FEM) or finite differences method (FDM), are most commonly used while solving field problems. Boundary methods, 'competitive' to them, such as the boundary elements method (BEM), are less popular in computation despite their advantageous features like simpler discretization, smaller size of numerical models, a possibility to analyse systems containing unlimited domains, easier solution error estimation, or its analytical form itself, to name a few. It might be so as the boundary methods are considered to be less universal due to being inapplicable to solving non-linear problems. Essentially, it seems right as all the boundary methods are based on superposition principle, which holds solely in linear problems. Nevertheless, certain concepts indicating their applicability to selected types of non-linear problems are known (see eg [1]).

The authors of the paper have long been involved in research and development of the fundamental solutions method (FSM) [2, 3, 4, 5], one of the methods from Treffz group, rarely applied to boundary electrodynamics problems. When compared to other boundary methods, its features such as lack of complicated and intricate analytical calculations, or lack of numerical computations to determine singular integrals (such as BEM) must be find attractive; it is also easier to choose base functions for the approximation sum as it simply requires defining a set of certain points called fundamental solutions singularities.

Moreover, by applying iterative FSM big equations systems to solve are avoided, computational time is definitely reduced, and concurrently to the iteration process the solution error can be controlled, which allows to stop computations upon achieving the required accuracy.

The paper is aimed at implementing iterative FSM to nonlinear problems analysis, with a model magnetostatics problem set as the testing example. Numerical test runs have been performed to estimate efficiency of the method and check feasibility of further research into its wider applying to more complex problems in electromagnetics.

General formulation of the problem

The problem under consideration deals with magnetostatic field analysis for the system presented in Figure 1, where Ω^{I} domain represents an solid of nonlinear magnetic properties ($\mu = \mu(H)$), immersed in an unlimited Ω^{II} domain with a constant magnetic permeability μ_0. The excitation sources of the magnetic field are direct currents of known density distribution $\boldsymbol{J}$, flowing in Ω^{III} domain. Under such assumption, within Ω^{I}, Ω^{II} domains free from current, magnetic field intensity $\boldsymbol{H}$ can be expressed as a scalar magnetic potential φ defined with $\boldsymbol{H} = -\operatorname{grad}\varphi$. As Gauss law provides for the magnetic field this potential shall satisfy the equations

(1) $$\operatorname{div}\left(\mu(H)\cdot\operatorname{grad}\varphi^{\mathrm{I}}\right)=0 \qquad \text{inside } \Omega^{\mathrm{I}}$$

(2) $$\Delta\varphi^{\mathrm{II}}=0 \qquad \text{inside } \Omega^{\mathrm{II}}$$

Due to identity

(3) $$\operatorname{div}\left(f\,\boldsymbol{v}\right)=\boldsymbol{v}\cdot\operatorname{grad}f+f\operatorname{div}\boldsymbol{v}$$

equation (1) takes the form

(4) $$\Delta\varphi^{\mathrm{I}}=-\frac{\operatorname{grad}\mu(H)\cdot\operatorname{grad}\varphi^{\mathrm{I}}}{\mu(H)}$$

Since at the boundary surface S (Fig. 1) continuity is required both for the tangent component of the magnetic field intensity and the normal component of magnetic induction, the potential function shall also satisfy the following boundary conditions:

(5) $$\varphi^{\mathrm{I}}\big|_{\mathrm{S}}=\varphi^{\mathrm{II}}\big|_{\mathrm{S}}$$

(6) $$\left.\frac{\partial\varphi^{\mathrm{I}}}{\partial n}\right|_{\mathrm{S}}=\mu_r(H)\left.\frac{\partial\varphi^{\mathrm{II}}}{\partial n}\right|_{\mathrm{S}}$$

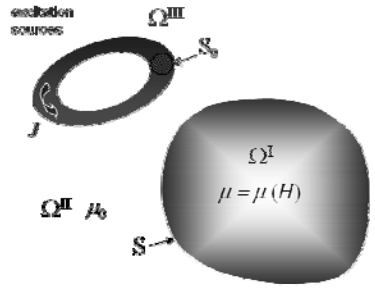

Fig. 1. The analysed system

Thus, the problem of computing the field in domains Ω^{I} and Ω^{II} has been practically reduced to finding two scalar functions φ^{I} and φ^{II}, which incorporate the excitation field and satisfy the equations (2) and (4) as well as the conditions (5) and (6).

The solution method

The problem defined above can be solved iteratively, with $\mu = \mathrm{const}$ at the first step, and subsequently a linear problem defined by equations (2), (4) and the boundary conditions (5), (6), whereas the right side of the equation (4) is determined from the field distribution computed at the previous step, which is treated as a function of space coordinates [1]. Hence, at the n^{th} iteration step a linear Poisson equation is solved, with the source function

$$f^{(n)} = \frac{\boldsymbol{H}^{(n-1)} \cdot \operatorname{grad} \mu^{(n)}}{\mu^{(n)}} \tag{7}$$

The equations (2) and (4) are solved in the next iteration steps by means of fundamental solutions method (FSM) [2,3,4,5]. The sought potential functions within the domains Ω^{I} and Ω^{II} are expressed as a superposition of the potential φ_0 representing the excitation field, the potential $\varphi_f^{(n)}$ related to $f^{(n)}$ function and a linear combination of the fundamental solutions to the Laplace equation, namely

$$\varphi^{K,(n)}(\mathrm{P}) = \varphi_0(\mathrm{P}) + \varphi_f^{(n)}(\mathrm{P}) + \sum_{m=1}^{M} p_m^K F_m^K(\mathrm{P}) \,, \; K = \mathrm{I},\, \mathrm{II}, \tag{8}$$

where:

$$\varphi_f^{(n)}(\mathrm{P}) = \frac{1}{4\pi} \iiint_{\Omega^{\mathrm{I}}} \frac{f^{(n)}}{r} \mathrm{d}\,\omega, \quad r = \left|\overrightarrow{\mathrm{QP}}\right|, \quad \mathrm{Q} \in \Omega^{\mathrm{I}} \tag{9}$$

$$F_m^K(\mathrm{P}) = \frac{1}{r_m}, \qquad r_m = \left|\overrightarrow{\mathrm{Q}_m\mathrm{P}}\right|, \qquad m = 1,..,M \tag{10}$$

p_m^K - coefficients of the approximation sum.

The singular points Q_m of the fundamental solutions F_m^{K} are defined beyond the domain, concerned with the specific solution, i.e. $\mathrm{Q}_m \in \Omega^{\mathrm{II}}$ for F_m^{I} and $\mathrm{Q}_m \in \Omega^{\mathrm{I}}$ for F_m^{II}. The selection method is arbitrary, though it is recommended to set them neither too distant to, nor too far from the boundary surface S, and distribute them around as regularly as possible.

Determining the original field potential φ_0 is a separate issue. For sources, where current lines are closed it can be determined from a general relationship [6,7]

$$\varphi_0(\mathrm{P}) = \frac{\mu}{4\pi} \iint_{\mathrm{S}_0} \Phi(\mathrm{P},\mathrm{Q}) \boldsymbol{J}(\mathrm{Q}) \cdot \mathrm{d}\,\boldsymbol{s}_0, \quad \mathrm{Q} \in \Omega^{\mathrm{III}} \tag{11}$$

where:

$$\Phi(\mathrm{P},\mathrm{Q}) = \iint_{\mathrm{A(Q)}} \frac{\boldsymbol{r}_0 \cdot \mathrm{d}\,\boldsymbol{a}}{{r_0}^3}, \qquad r_0 = \overrightarrow{\mathrm{QP}} \tag{12}$$

Φ - the solid angle at which A surface is seen; it is set upon an infinitely narrow current stream passing Q point of the Ω^{III} domain, the remaining items denoted as in the Figure 1. It should be stressed that the method might prove divergent. It means some uncertainty exists whether the solution obtained in the next iteration step is burdened with a smaller error than the previous one. Thus, it is vital to be able to estimate properly the solution error at each iteration. The procedure presented further, based on the iterative FSM facilitates such estimation.

Computing coefficients of the approximation sum

By applying the best approximation method [8] all the p_m^K coefficients in (8) for the specific set of Q_m singular points can be optimally computed, i.e. the boundary error of the solution can be minimised. For the considered case the error has been determined as

$$\varepsilon = \sqrt{w_1 \varepsilon_1^2 + w_2 \varepsilon_2^2}, \quad w_1 + w_2 = 1 \tag{13}$$

where

$$\varepsilon_1 = \frac{1}{\varphi_{\max}} \sqrt{\frac{1}{S} \iint_{\mathrm{S}} \delta_1^2 \,\mathrm{d}\,s}, \quad \varepsilon_2 = \frac{1}{H_{\max}} \sqrt{\frac{1}{S} \iint_{\mathrm{S}} \delta_2^2 \,\mathrm{d}\,s} \tag{14}$$

$$\delta_1 = \left(\varphi^{\mathrm{II}} - \varphi^{\mathrm{I}}\right)\Big|_{\mathrm{S}}, \qquad \delta_2 = \left(\frac{\partial \varphi^{\mathrm{II}}}{\partial n} - \mu_r \frac{\partial \varphi^{\mathrm{I}}}{\partial n}\right)\Big|_{\mathrm{S}} \tag{15}$$

w_1, w_2 – weight coefficients, fitted experimentally to optimise convergence

$\varphi_{\max}$, $H_{\max}$ – maximum values for the potential and magnetic field intensity at the boundary surface S

δ_1, δ_2 – determined at the surface S boundary error functions (5) and (6), respectively.

Still, the results of many numerical experiments [3,4,5] indicate that it is quicker and significantly simpler to set up a iteration procedure where at each iteration step only one p_m^K coefficient for each of system domains. In such cases error minimalisation conditions (13)

$$\frac{\partial \varepsilon}{\partial p_m^K} = 0, \quad K = \mathrm{I}, \mathrm{II} \tag{16}$$

result in a system of 2nd order linear equations, which can be solved explicitly

$$p_m^{\mathrm{I}} = \frac{A_{22}B_1 - A_{12}B_2}{A_{11}A_{22} - A_{12}A_{21}}, \qquad p_m^{\mathrm{II}} = \frac{A_{11}B_2 - A_{21}B_2}{A_{11}A_{22} - A_{12}A_{21}} \tag{17}$$

where:

$$\begin{aligned}
A_{11} &= \frac{1}{\varphi_{\max}^2} \iint_{\mathrm{S}} \left(\varphi^{\mathrm{I}}\right)^2 \mathrm{d}s + \frac{\mu_r^2}{H_{\max}^2} \iint_{\mathrm{S}} \left(\frac{\partial \varphi^{\mathrm{I}}}{\partial n}\right)^2 \mathrm{d}\,s \\
A_{12} &= -A_{21} = \frac{1}{\varphi_{\max}^2} \iint_{S} \varphi^{\mathrm{I}} \varphi^{\mathrm{II}} \,\mathrm{d}\,s + \frac{\mu_r^2}{H_{\max}^2} \iint_{S} \frac{\partial \varphi^{\mathrm{I}}}{\partial n} \frac{\partial \varphi^{\mathrm{II}}}{\partial n} \mathrm{d}\,s \\
A_{22} &= \frac{1}{\varphi_{\max}^2} \iint_{\mathrm{S}} \left(\varphi^{\mathrm{II}}\right)^2 \mathrm{d}\,s + \frac{\mu_r^2}{H_{\max}^2} \iint_{\mathrm{S}} \left(\frac{\partial \varphi^{\mathrm{II}}}{\partial n}\right)^2 \mathrm{d}\,s
\end{aligned} \tag{18}$$

$$\begin{aligned}
B_1 &= \frac{1}{\varphi_{\max}^2} \iint_{\mathrm{S}} \varphi^{\mathrm{I}} \delta_1^{(m-1)} \mathrm{d}s + \frac{\mu_r}{H_{\max}^2} \iint_{\mathrm{S}} \frac{\partial \varphi^{\mathrm{I}}}{\partial n} \frac{\partial \delta_2^{(m-1)}}{\partial n} \mathrm{d}s \\
B_2 &= \frac{1}{\varphi_{\max}^2} \iint_{\mathrm{S}} \varphi^{\mathrm{II}} \delta_1^{(m-1)} \mathrm{d}s + \frac{1}{H_{\max}^2} \iint_{\mathrm{S}} \frac{\partial \varphi^{\mathrm{II}}}{\partial n} \frac{\partial \delta_2^{(m-1)}}{\partial n} \mathrm{d}s
\end{aligned} \tag{19}$$

That eliminates time consuming numerical solving of big equation systems, which significantly reduces real computation time; resulting longer approximation sums in (8) are fortunately practically immaterial.

The rudimentary approximation theorem on existence and uniqueness of the linear approximation solution [8] it can be proven [2,3] that at the m^{th} iteration step

(20) $$\varepsilon^{(m)} \le \varepsilon^{(m-1)}$$

which means that the procedure converges for the linear problem. Additionally, it provides opportunities to solution error control in the course of iteration process, which may be automatically terminated once the required accuracy has been achieved (see remarks at the end f the previous section).

The algorithm

The proposed procedure, as described herein, includes two iterative loops, namely an internal one aimed at obtaining linear problem solution within the set accuracy, and the external one whose subsequent steps modify the right side function of the equation (4) in dependence from the magnetic field distribution computed at the preceding step. The simplified algorithm of the procedure is illustrated in Figure 2.

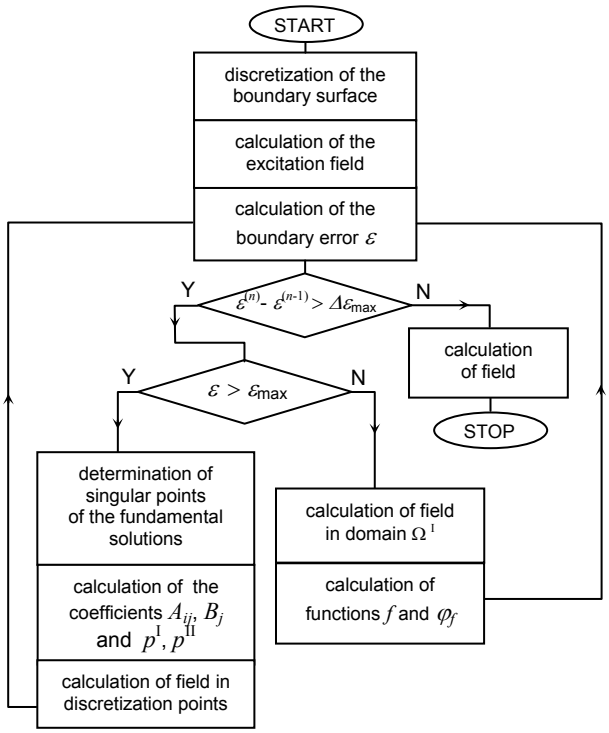

Fig. 2. The method's algorithm

Test

To estimate the correctness and efficiency of the presented method a model system for determining the magnetostatic field has been considered (see Figure 3). The original source of the field is the direct current i flowing in a circular coil, which surrounds a spherical solid of non-linear magnetic properties.

The assumed $\mu(H)$ dependence was that of ST35 steel (based on [9]) – see Figure 4.

The original field potential calculated from (11) and (12) is expressed as

(21) $$\varphi_1^0 = -\frac{i}{2\pi} z \int_0^{\pi} \frac{\rho R_u \cos\theta - \rho^2 - z^2}{(\rho^2 \sin^2\theta + z^2)\sqrt{R_u^2 - 2\rho R_u \cos\theta + \rho^2 + z^2}} \mathrm{d}\theta$$

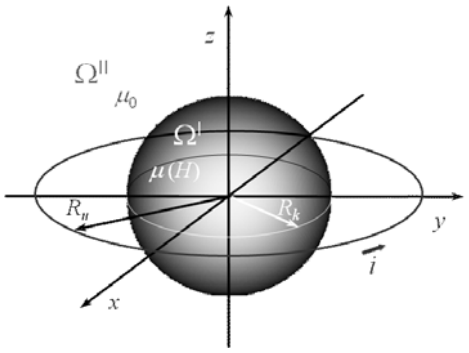

Fig. 3. Model system under analysis

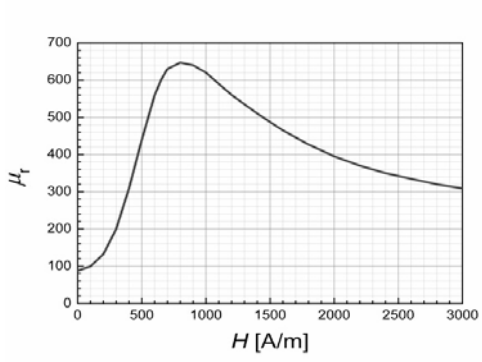

Fig. 4. $\mu(H)$ dependence (steel ST35, [9])

The proposed algorithm was converted into a Fortran77 computer routine and then several numerical test runs were performed. Singular points for the fundamental Q_m solution are selected randomly at each iteration step by means of pseudo-random number generator, within the user defined distance from the boundary surface S. Exemplary results are presented in diagrams in Figures 5-9. They concern the system with the following parameters: i = 0.5 kA, R_u = 0.3 m, R_k = 0.1 m.

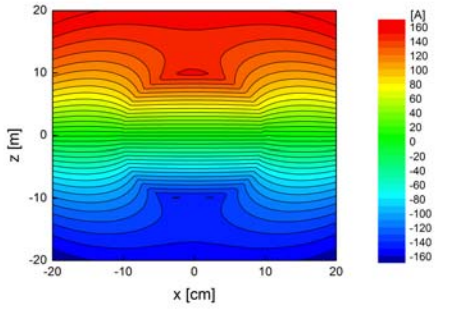

Fig. 5. Computed magnetic potential distribution for the y = 0 plane

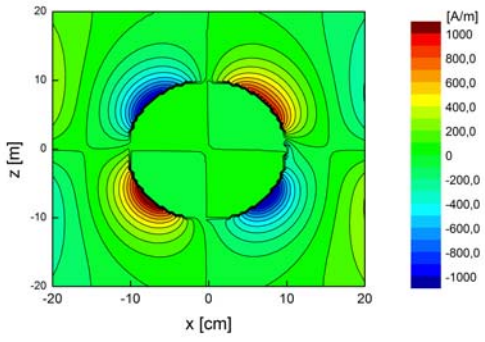

Fig. 6. Computed distribution H_x for the y = 0 plane

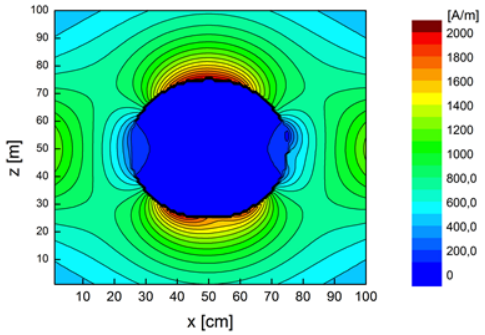

Fig. 7. Computed distribution H_z for the y = 0 plane

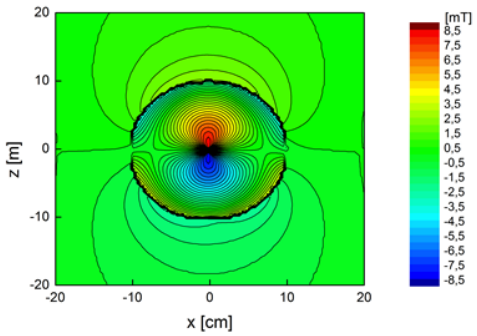

Fig. 8. Computed distribution for the radial component of the magnetic field intensity for the y = 0 plane

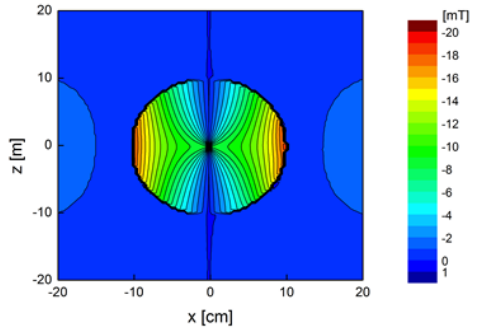

Fig 9. Computed distribution for the zenith component of the magnetic field intensity for the y = 0 plane

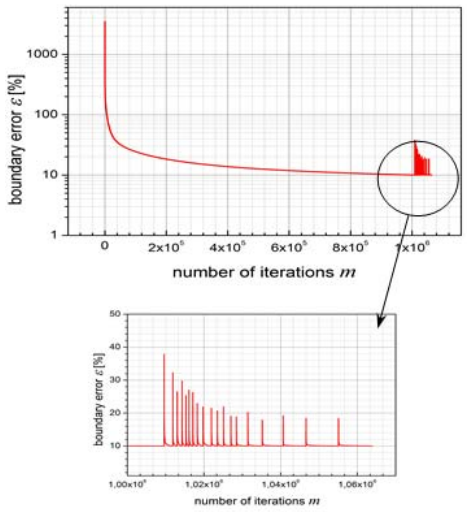

Fig. 10. Boundary error of the solution (see (13), (14)) at subsequent iteration steps

Summary

The method for solving non-linear 3D magnetostatics problems is presented in the paper. It is based on the iterative version of the fundamental solutions method. A dozen of numerical tests performed on a simple model system provided grounds to evaluate the method as effective and correct. It was found that the method divergence rate majorly depends on the original field and the relative magnetic permeability (the greater mean values these parameters gain within Ω^I domain, the slower the iteration process divergence is). Though the obtained results are found prospective, the complete evaluation of its applicability require further research into more complex systems modelling real technical systems.

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***Authors**: dr hab. Stanisław Pawłowski, Department of Electrodynamics and Electrical Machine Systems, al. Powstańców Warszawy 12, 35-959 Rzeszów, E-mail: spawlo@prz.edu.pl, dr inż. Jolanta Plewako, Department of Power Electronics and Power Engineering, al. Powstańców Warszawy 12, 35-959 Rzeszów, E-mail: jplewako@prz.edu.pl.*