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The Shape Reconstruction of Unknown Objects for Inverse Problems

***Abstract**. The proposed solution of the inverse problem in the Electrical Impedance Tomography was based on a numerical scheme for the identification of the piecewise constant conductivity. The level set method is a powerful tool for representing moving of stationary interfaces. The representation of the shape of the boundary and its evolution during an iterative reconstruction process is achieved by the level set function. The forward problem was solved by the finite element method. The iterative algorithms are based a combination of the these methods.*

***Streszczenie.** Proponowane rozwiązanie zagadnienia odwrotnego w tomografii impedancyjnej zostało oparte na algorytmach numerycznych identyfikujących obiekty o różnych konduktywnościach. Reprezentację kształtu brzegu oraz jego ewolucję podczas procesu rekonstrukcji opisuje metoda zbiorów poziomicowych. Zagadnienie proste zostało rozwiązane za pomocą metody elementów skończonych. Rozwiązanie zagadnienia odwrotnego oparte zostało na odpowiednim złożeniu wymienionych metod. (**Rekonstrukcja kształtu nieznanych obiektów w zagadnieniach odwrotnych**).*

Keywords: Electrical Impedance Tomography, Level Set Methods, Inverse Problem
Słowa kluczowe: tomografia impedancyjna, metoda zbiorów poziomicowych, zagadnienie odwrotne

Electrical Impedance Tomography

In this paper was proposed the topological numerical technique with different advantages to solve the inverse problem in the electrical impedance tomography (EIT) [2,7-9]. The level set method is known to be a powerful and versatile tool to model the evolution of interfaces [1,3,4]. The idea is merely to define a smooth function ϕ, that represents the interface and has the following properties (fig.1):

$$\begin{aligned} &\phi(x,t) > 0 \text{ for } x \in \Omega \\ &\phi(x,t) < 0 \text{ for } x \notin \Omega \\ &\phi(x,t) = 0 \text{ for } x \in \partial\Omega = \Gamma(t) \end{aligned} \tag{1}$$

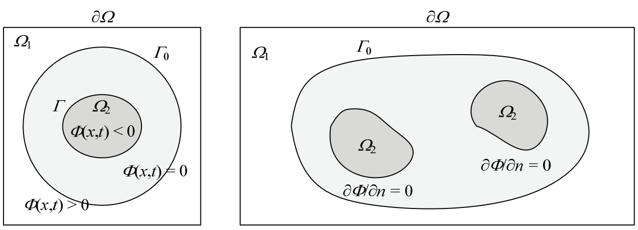

Fig. 1. The representation of the level set function Φ(x,t) (where:Ω_1-region, Ω_2-subregions, Γ-interface of the level set function, Γ_o-the zero level set function, ðΩ-the outside area)

Thus, the interface is captured for all later time, by localization of the set Γ(t) for which ϕ vanishes. This deceptively trivial statement is of great significance for numerical computation primarily, because topological changes such as breaking and merging are well defined and performed. The motion is analyzed by the convection the ϕ values (levels) with the velocity field $\mathbf{v}$. The Hamilton-Jacobi equation of the form [6]

$$\frac{\partial\phi}{\partial t} + \mathbf{v} \cdot \nabla\phi = 0 \tag{2}$$

is describing this process.

Here $\mathbf{v}$ is the desired velocity on the interface, and is arbitrary elsewhere. Actually, only the normal component of $\mathbf{v}$ is needed $\mathbf{v}_{\mathrm{N}} = \mathbf{v} \cdot \frac{\nabla\phi}{|\nabla\phi|}$, so (2) becomes

$$\frac{\partial\phi}{\partial t} + \mathbf{v}_N \cdot |\nabla\phi| = 0 \tag{3}$$

In the level set representation, the interface, which is the set of points (*x,y*) satisfying $\phi(x,t) = 0$ is not explicitly given. There is only information $\phi(x_i, y_i)$ at each grid point.

The expression for the curvature of the zero level set assigned to the interface itself is given by:

$$\kappa = \nabla \cdot \frac{\nabla\phi}{|\nabla\phi|} = \frac{\phi_{xx}\phi_y^2 - 2\phi_y\phi_x\phi_{xy} + \phi_{yy}\phi_x^2}{\left(\phi_x^2 + \phi_y^2\right)^{3/2}} \tag{4}$$

Update the level set function ϕ (*x,t*) by solving the Hamilton-Jacobi equation:

$$\frac{\phi^{k+1} - \phi^k}{\Delta t} + \mathbf{v}_k \left|\nabla\phi^k\right| = 0 \tag{5}$$

Transforming above equation:

$$\phi^{k+1} = \phi^k - \Delta t\, \mathbf{v}_k \left|\nabla\phi^k\right| \tag{6}$$

The gradient of the level set function $\left|\nabla\phi^k\right|$ in following iterations was calculated by the scheme ENO (*essentially non-oscillatory polynomial interpolation* [5]. The stability of received solution is guaranteed by Couranta-Friedreichsa-Levy'ego condition (*CFL condition*). The CFL is described as:

$$\Delta t < \frac{\Delta x}{\max\{|\mathbf{v}_n|\}} \tag{7}$$

The equation is enforced by choosing a CFL number α which the optimum value is 0,9.

(8) $$\Delta t\left(\frac{\max\{|\nu_n|\}}{\Delta x}\right)=\alpha$$

and $0<\alpha<1$.

When flat or steep regions complicate the determination of the contour reinitialization is necessary.
The level set function is signed distance function:

(9) $$|\nabla\phi|=1$$

Reinitialization is based by replacing $\boldsymbol{\phi}$ by another function that has the same zero level set but behaves better. This is based on following partial differential equation:

(10) $$\frac{\partial}{\partial t}\phi+S(\phi)(\nabla\phi-1)=0$$

where S($\boldsymbol{\phi}$) is defined as:

(11) $$S(\phi)=\begin{cases}-1 & \text{for } \phi<0\\ 0 & \text{for } \phi=0\\ 1 & \text{for } \phi>0\end{cases}$$

This partial differential equation is solved until a steady state is achieved. Similar to the velocity extension a first order upwind scheme for the spatial and a forward Euler time discretization is used.

(12)
$$\phi^{k+1}=\phi^{k}-\frac{\Delta t}{\Delta x}S^{+}(\sqrt{\max[(a^{+})^{2},(b^{-})^{2}]+\min[(c^{+})^{2},(d^{-})^{2}]}-1)$$
$$-\frac{\Delta t}{\Delta x}S^{-}(\sqrt{\max[(a^{-})^{2},(b^{+})^{2}]+\min[(c^{-})^{2},(d^{+})^{2}]}-1)$$

$$(\cdot)^{+}=\max(\cdot,0)\quad(\cdot)^{-}=\min(\cdot,0)$$

a, b, c, d are defined following:

$$a=D_x^{-}\phi_{ij}^{k},\quad b=D_x^{+}\phi_{ij}^{k},\quad c=D_y^{-}\phi_{ij}^{k},\quad d=D_y^{-}\phi_{ij}^{k}$$

The forward problem in EIT is solving by Laplace's equation:

(13) $$\mathrm{div}(\gamma\,\mathrm{grad}\,\mathbf{u})=0$$

where **u** - electric potential, **γ** - conductivity.
The following functional is minimized:

(14) $$F=0.5\sum_{j=1}^{p}(\mathbf{U}-\mathbf{U_0})^{T}(\mathbf{U}-\mathbf{U_0})$$

where p is the number of the projection angles.
The following steps are used in numerical algorithm:

- From the level set function $\phi(x,y)$ (initial) at a time level t, find necessary interface information
$\Gamma_o=\{\phi(x,y)=0\}$
- Calculate the electric potential (solving the Laplace equation by using Finite Element Method)
$-\Delta\mathbf{u}=0$
- Compute the difference of the computed solution with the observed data
$\mathbf{u}(\Gamma_k)-\mathbf{u}_0$
- Solve the adjoint equation
$-\Delta\mathbf{p}=\mathbf{u}-\mathbf{u}_0$
- Find the component of the normal velocity of the surface due to the electric potential
- Find the normal velocity of the level set function
- Calculate the velocity
$\nu_k=\nabla\mathbf{p}_k\cdot\nabla\mathbf{u}_k$
- Update the level set function
- Reinitialize the level set function

Results

The pictures show a few of the practical examples (the tree trunks, the copper-mine ceiling, the moisture wall) with different objects and their process reconstruction. The images show the original object and the reconstructed shape after the process iterations. The figure 2 presents an image reconstruction in the electrical impedance tomography of the trunk tree. The objective functions of the object were shown on figure 3. The figures 4 and 5 present the image reconstruction and objective function of the copper-mine ceiling. The pictures present different objects and the process reconstruction. The original object is noted by the blue line and the final figure is pink. The reconstruction is good, because the region borders nearly are located the object edges. Using different the zero level set function in the moisture wall (fig. 6), the object function achieves the minimum after the various number of iterations. The object in the figure achieves the minimum after 52 iterations. The shape and size of the zero level set function does not influence on the final form of the finding object.

a) B)

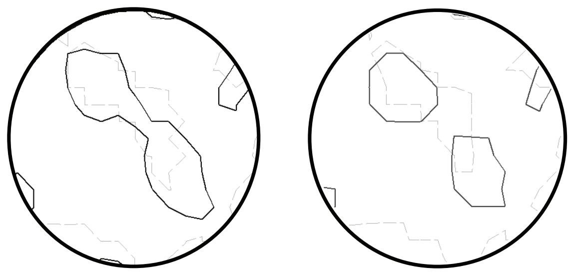

Fig. 2. The image reconstruction of the tree trunks, a) without regularization b) with regularization

a)

b)

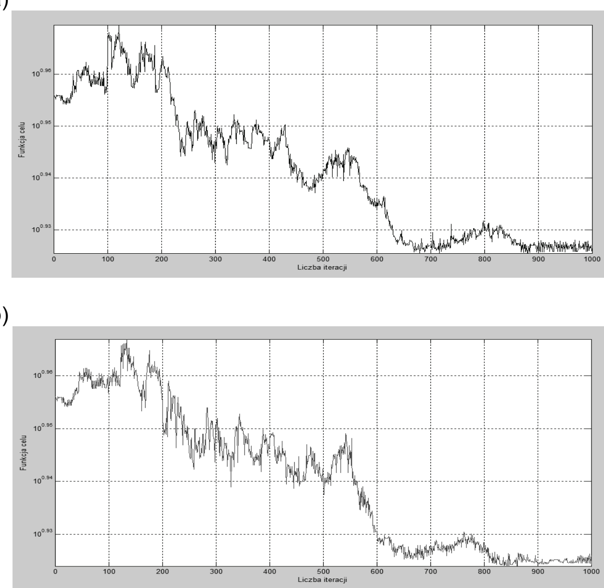

Fig.3. The objective function of the tree trunks, a) without regularization b) with regularization

a)

b)

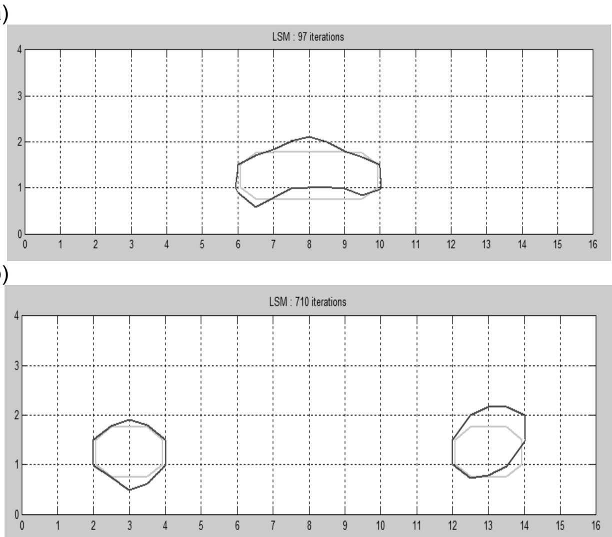

Fig. 4. The image reconstruction: of the copper-mine ceiling: a) one object, b) two objects

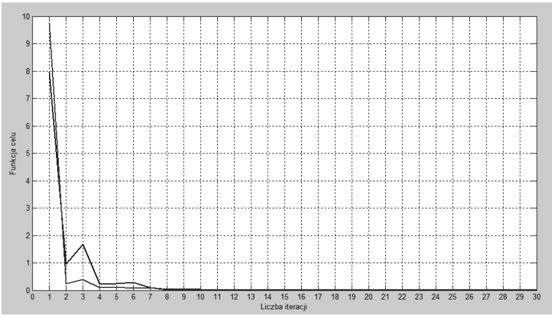

Fig. 5. The objective function of the copper-mine ceiling

a)

b)

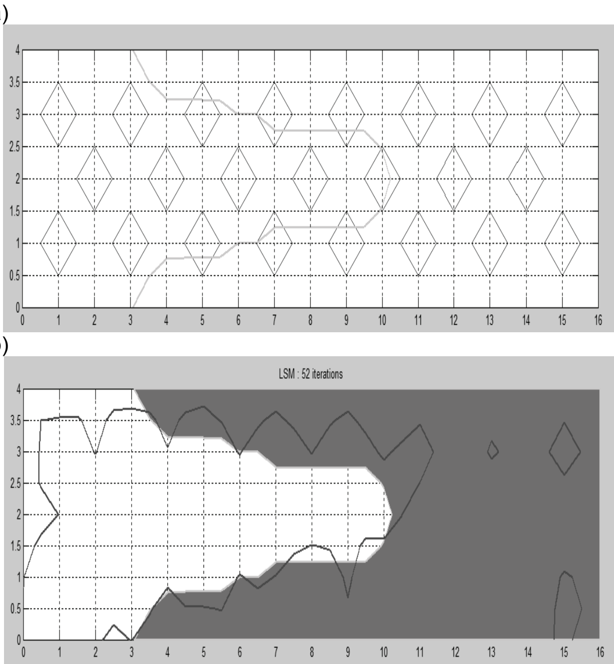

Fig. 6. The image reconstruction of the moisture wall, a) the initial level set function, b) the image reconstruction

Summary

This paper has introduced the new method of approximation of material coefficient. There was discussed the application of the level set function for identifying the unknown shape of an interface in a problem motivated by electrical impedance tomography. The pictures show a few of the practical examples such as the tree trunks, the copper-mine ceiling, the moisture wall with different objects and their process reconstruction. The level set function techniques was shown to be successful to identify the unknown boundary shapes.

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