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Robust H∞ output feedback control for constrained discrete-time piecewise affine systems with time-parametric uncertainties

***Abstract**. In this study, a novel robust H∞ output feedback control scheme is presented for discrete-time piecewise affine (PWA) systems in the presence of time-varying uncertainties, external disturbance and time-domain constraints. The suggested control method is formulated as linear matrix inequalities (LMIs). The basic idea of them is to construct piecewise quadratic Lyapunov function and introduce a dissipation inequality to guarantee the system energy dissipation. The designed controllers not only guarantee the stability of the closed-loop systems, but also obtain the disturbance attenuation ability.*

***Streszczenie.** Zaprezentowano nowy odporny układ sterowania ze sprzężeniem typu H∞ do systemów PWA. Uwzględniono obecność zmiennych w czasie niepewności, zewnętrznych zakłóceń i czasowo zależnych wymuszeń.(**Odporny układ sterowania H∞ dla systemów z czasowo zależnymi niepewnościami**)*

Keywords: piecewise affine systems, linear matrix inequality approach, static output feedback
Słowa kluczowe: systemy affiniczne (pokrewne), sprzężenie zwrotne, sterowanie, stabilność.

Introduction

Piecewise affine (PWA) systems have been receiving increasing attention by the control community in the recent years because they provide a powerful means of analysis and design for non-linear control systems [1-3]. In fact, piecewise linear systems constitute a special class of hybrid systems and arise often in practical control systems when piecewise linear components, such as deadzone, saturation, relays, and hysteresis, are encountered.

A number of results have been obtained on analysis and controller design of such PWA systems during the last few years. During the last decade, several output feedback controller design methods have be developed for PWA systems based on piecewise-quadratic Lyapunov function (PWQLF)[4]. There is a common restriction in aforementioned papers, i.e. the control design techniques provide controllers of order equal to the plant order. During the last decade, in the framework of linear matrix inequalities (LMIs) [5], lots of results on stability and H1 performance analysis have been developed for both continuous-time and discrete-time PWA systems [6–13]. The SOF problem for PWA systems has received less attention in the literature because the synthesis of a static output feedback gain is difficult. Hence, the objective of this paper is to suggest a H_∞ SOF control problem for discrete-time PWA systems.

Problem Statement

Consider the discrete-time PWA system of the form

$$
\begin{aligned}
&x(k+1)=A_{si}(k)x(k)+B_{s1i}(k)w(k)+B_{s2i}(k)u(k)+b_{si}(k)\\
&z(k)=C_{si}(k)x(k)+D_{si}(k)u(t)\\
&y(k)=C_{yi}(k)x(k)\\
&\mathit{for}\quad y(k)\in S_i, i\in \wp
\end{aligned}
\tag{1}
$$

Subject to time-domain constraints

$$
|u_i(k)|\le u_{i,\max},\qquad i=1,2,\cdots,m \tag{2}
$$

where $\{S_i\}_{,i\in\wp}\subseteq R^p$ denotes a partition of the output space into a number of closed polyhedral regions, $\wp$ is the index set of these regions, $x(k)\in R^n$ is the state, $u(t)\in R^m$ is the input, $y(t)\in R^p$ is the measured output, $b_i(k)$ is the affine term. The i th local model of the system satisfies

$$
\begin{bmatrix} A_{si}(k) & B_{s1i}(k) & B_{s2i}(k) & b_{si}(k)\\ C_{si}(k) & 0 & D_{si}(k) & 0\\ C_{yi}(k) & 0 & 0 & 0\end{bmatrix}=
\sum_{l_i=1}^{L_i}\xi_{il_i}(k)\begin{bmatrix} A_{sil_i} & B_{s1il_i} & B_{s2il_i} & b_{sil_i}\\ C_{sil_i} & 0 & D_{sil_i} & 0\\ C_{yil_i} & 0 & 0 & 0\end{bmatrix},
\tag{3}
$$

$$
\xi_{il_i}(k)\ge 0,\ \sum_{l_i=1}^{L_i}\xi_{il_i}(k)=1
$$

where $\xi_{il_i}(k)$ is an unknown time-varying parameter. $\wp$ is partitioned as $\wp=\wp_0\bigcup\wp_1$, where $\wp_0$ is the index set of the subspaces that contain the origin, that is $b_i(k)=0$, and $\wp_1$ is the index set of the regions otherwise. For future use, defined a set Ω that represents all possible transitions from one region to itself or another region, that is

$$
\Omega=\left\{(i,j)\middle| x(k)\in S_i, x(k+1)\in S_j\right\}
$$

There are there basic assumptions in this paper:

(AI) when the system transits from the region S_i to S_j at time k, the dynamics of the system is governed by the dynamics of the local model of S_i at that time.

(A2) Matrix E_i and scalar f_i exist such that $S_i\in\varepsilon_i$ where $\varepsilon_i=\{y(k)\ \ \|E_i y(k)+f_i\|\le 1\}$. The ellipsoid ε_i can also be described in the form of

$$
\begin{bmatrix} y(k)\\ 1\end{bmatrix}^T\begin{bmatrix} E_i^T E_i & *\\ f_i^T E_i & f_i^T f_i-1\end{bmatrix}\begin{bmatrix} y(k)\\ 1\end{bmatrix}\le 0 \tag{4}
$$

(A3) As the matrix C_{yi} is full row rank, i.e. rank (C_{yi})= p_i , there exists a nonsingular matrix R_i such that $C_i=C_{yi}R_i^{-1}=\begin{bmatrix} I_{pi} & 0\end{bmatrix}$.

The problem to be addressed in this paper is to design a controller for (1) such that the closed-loop system is asymptotically stable, and the H_∞ norm from the disturbance w to the performance output z is minimised, while the time-domain constraints (2) are respected. In this paper, the controller with the following structure is considered

(5) $$u(k) = L_i y(k)$$

then the closed-loop system admits the realization

(6) $$x(k+1) = A_{cli}(k)x(k) + B_{1i}(k)w(k) + b_i(k)$$
$$z(k) = C_{cli}(k)x(k)$$

Where: $A_{cli}(k) = A_{si}(k) + B_{s2i}(k)L_i C_{yi}(k)$ and $C_{cli} = C_i(k) + D_i(k)L_i C_y(k)$.

Robust unconstrained H_∞ control

The time-domain constraints (2) are neglected and an LMI based robust unconstrained H_∞ controller is designed in this section. At first, some preliminary results are presented in the following.

(7)
$$\begin{bmatrix} Q_{il_i1} & * & * & * & * & * & * \\ 0 & S_{il_i} & * & * & * & * & * \\ 0 & 0 & \gamma^2 I & * & * & * & * \\ \begin{matrix} A_{11il_i}Q_{il_i1} + A_{12il_i}Q_{il_i2}^T \\ +B_{21il_i}Y_{il_i} \end{matrix} & A_{12il_i}S_{il_i} & B_{1il_i} & Q_{jl_j1} + \alpha_i b_{1il_i} b_{1il_i}^T & * & * & * \\ \begin{matrix} A_{21il_i}Q_{il_i1} + A_{22il_i}Q_{il_i2}^T \\ +B_{22il_i}Y_{il_i} \end{matrix} & A_{22il_i}S_{il_i} & B_{2il_i} & Q_{jl_j2} + \alpha_i b_{2il_i} b_{1il_i}^T & Q_{j3} + \alpha_i b_{2il_i} b_{2il_i}^T & * & * \\ E_{1il_i}Q_{il_i1} + E_{2il_i}Q_{il_i2}^T & E_{2il_i}S_{il_i} & 0 & \alpha_i f_i b_{1il_i}^T & \alpha_i f_i b_{2il_i}^T & \alpha_i (f_i f_i^T - 1) & * \\ C_{1il_i}Q_{il_i1} + C_{2il_i}Q_{il_i2}^T & C_{2il_i}S_{il_i} & 0 & 0 & 0 & 0 & I \end{bmatrix} \geq 0$$

For $i \in \wp_i$

(8)
$$\begin{bmatrix} Q_{il_i1} & * & * & * & * & * \\ 0 & S_{il_i} & * & * & * & * \\ 0 & 0 & \gamma^2 I & * & * & * \\ A_{11il_i}Q_{il_i1} + A_{12il_i}Q_{il_i2}^T + B_{21il_i}Y_{il_i} & A_{12il_i}S_{il_i} & B_{1il_i} & Q_{jl_j1} + \alpha_i b_{1il_i} b_{1il_i}^T & * & * \\ A_{21il_i}Q_{il_i1} + A_{22il_i}Q_{il_i2}^T + B_{22il_i}Y_{il_i} & A_{22il_i}S_{il_i} & B_{2il_i} & Q_{jl_j2} + \alpha_i b_{2il_i} b_{1il_i}^T & Q_{j3} + \alpha_i b_{2il_i} b_{2il_i}^T & * \\ C_{1il_i}Q_{il_i1} + C_{2il_i}Q_{il_i2}^T & C_{2il_i}S_{il_i} & 0 & 0 & 0 & I \end{bmatrix} \geq 0$$

for $i \in \wp_0$

where the symbol "*"represents the transpose of the related term. The associated controller matrices can be computed by

(9) $$u(k) = \sum_{l_i=1}^{L_i} \xi_{il_i}(k) L_{il_i}, \quad L_{il_i} = Y_{il_i} Q_{il_i1}^{-1}$$

This can be briefly shown as follows.

It is assumed that there exists a PWQLF as $V(t) = x^T(k)\overline{P}_i x(k)$ with $\overline{P}_i(k) = \sum_{l_i=1}^{L_i} \xi_{il_i}(k)\overline{P}_{il_i}$, $\overline{P}_i = \overline{P}_i^T > 0$ satisfying the dissipation inequality

(10) $$x(k+1)^T \overline{P}_j x(k+1) + \|z(k)\|^2 \leq \gamma^2 \|w(k)\|^2 + x(k)^T \overline{P}_i x(k)$$

The above inequality implies

(11) $$V(x(k+1)) + \sum_{g=0}^{k-1} \|z(g)\|^2 \leq \gamma^2 \sum_{g=0}^{k-1} \|w(g)\|^2 + V(x(0))$$

For all $k \geq 0$. Owing to $V(x(k)) \geq 0$, with $x(0) = 0$, we have

(12) $$\sum_{g=0}^{\infty} \|z(g)\|^2 \leq \gamma^2 \sum_{g=0}^{\infty} \|w(g)\|^2$$

From which it is concluded that $\|z(k)\|^2 < \gamma^2 \|w(k)\|^2$, that is the H_∞ norm from the disturbance w to the performance output z is less than γ.

In the following, our interest is to seek conditions in the form of LMIs for guaranteing (10). To treat the affine therm $b_i(k)$, the tegion information Eq. 4 of the controller is taken into account. By the S-procedure[5], Eq. 10 is satisfied if there exists $\lambda_i > 0$ such that

$$(11)\quad \begin{bmatrix} x(k) \\ w(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} A_{cli}^T(k)\bar{P}_j A_{cli}(k) - \bar{P}_i + C_{cli}^T(k)C_{cli}(k) & A_{cli}^T(k)\bar{P}_j B_{1i}(k) & A_{cli}^T(k)\bar{P}_j b_i(k) \\ * & B_{1i}^T(k)\bar{P}_j B_{1i}(k) - \gamma^2 I & B_{1i}^T(k)\bar{P}_j b_i(k) \\ * & * & b_i^T(k)\bar{P}_j b_i(k) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ 1 \end{bmatrix} - \lambda_i \begin{bmatrix} y(k) \\ 1 \end{bmatrix}^T \begin{bmatrix} E_i^T E_i & * \\ f_i^T E_i & f_i^T f_i - 1 \end{bmatrix} \begin{bmatrix} y(k) \\ 1 \end{bmatrix} < 0$$

The sufficient condition of Eq. 11 is that

$$(12)\quad \begin{bmatrix} \bar{P}_i & 0 & A_{cli}^T(k) & C_{yi}^T(k)E_i^T & C_{cli}^T \\ * & \gamma^2 I & B_{1i}^T(k) & 0 & 0 \\ * & * & \bar{P}_j^{-1} + \lambda_i^{-1} b_i b_i^T & \lambda_i^{-1} b_i f_i^T & 0 \\ * & * & * & \lambda_i^{-1}(f_i f_i^T - 1) & 0 \\ * & * & * & * & I \end{bmatrix} > 0$$

Lemma1: Conditon (12) is satisfied if

$$(13)\quad \begin{bmatrix} \bar{P}_{iL_i} & 0 & A_{clil_i}^T(k) & C_{yil_i}^T(k)E_{il_i}^T & C_{clil_i}^T \\ * & \gamma^2 I & B_{1il_i}^T(k) & 0 & 0 \\ * & * & \bar{P}_{jl_j}^{-1} + \lambda_i^{-1} b_{il_i} b_{il_i}^T & \lambda_i^{-1} b_{il_i} f_i^T & 0 \\ * & * & * & \lambda_i^{-1}(f_i f_i^T - 1) & 0 \\ * & * & * & * & I \end{bmatrix} > 0$$

for $i \in \wp_i$

$$(14)\quad \begin{bmatrix} \bar{P}_{il_i} & 0 & A_{clil_i}^T(k) & C_{clil_i}^T \\ * & \gamma^2 I & B_{1il_i}^T(k) & 0 \\ * & * & \bar{P}_{jl_j}^{-1} + \lambda_i^{-1} b_{il_i} b_{il_i}^T & 0 \\ * & * & * & I \end{bmatrix} \ge 0 \quad \text{for } i \in \wp_0$$

Proof: At first, fix the index pair (i, j) and assume that conditon (12) is feasible. For each l_i , multiply the corresponding $l_j = 1,\cdots,L_j$ inequalities by $\xi_{jl_j}(k+1)$ and sum. Then multiply the corresponding $l_j = 1,\cdots,L_j$ inequalitiexby $\xi_{jl_i}(k)$ and sum, (13) can be achieved. In the same way , (14) can by achieved from (14)

By pre- and post-multiplying Eq.13 via $diag\{\bar{P}_{il_i}^{-1} \quad I \quad I \quad I \quad I\}$, Now, apply the congruence transformation $diag\left(R_{il_i}, I, R_{il_i}, I, I\right)$ to matrix inequalities Eq.13, respectively. We can get

$$\begin{bmatrix} Q_{il_i} & * & * & * & * \\ 0 & \gamma^2 I & * & * & * \\ (A_{sil_i} + B_{s2il_i} L_{il_i} C_{yil_i} R_{il_i}^{-1})Q_{il_i} & B_{s1il_i} & Q_{jl_j} + \alpha_i b_{sil_i} b_{sil_i}^T & * & * \\ E_{il_i} Q_{il_i} & 0 & \alpha_i f_i b_{il_i}^T & \alpha_i (f_i f_i^T - 1) & * \\ (C_{sil_i} + D_{sil_i} L_{il_i} C_{yl_i} T_{il_i}^{-1})Q_{il_i} & 0 & 0 & 0 & I \end{bmatrix} > 0 \quad (15)$$

where : $\bar{P}_{il_i}^{-1} = \bar{Q}_{il_i}, \bar{P}_{jl_j}^{-1} = \bar{Q}_{jl_j}, \lambda_i^{-1} = \alpha_i$. $Q_{il_i} = R_{l_i} \bar{Q}_{il_i} R_{l_i}^T, A_{il_i} = R_{l_i} A_{sil_i} R_{l_i}^{-1}, B_{1il_i} = R_{l_i} B_{s1il_i}, B_{2il_i} = R_{l_i} B_{s2il_i}$ $b_{il_i} = R_{l_i} b_{il_i}, E_{il_i} = R_{l_i} E_{il_i} R_{l_i}^{-1}, C_{il_i} = C_{il_i} R_{l_i}^{-1}$.

$S_{il_i} = Q_{il_i 3} - Q_{il_i 2}^T Q_{il_i 1}^{-1} Q_{il_i 2} > 0$ using the Schur complement. The matrices $A_{il_i} = \begin{bmatrix} A_{11il_i} & A_{12il_i} \\ A_{21il_i} & A_{22il_i} \end{bmatrix}$, $B_{1il_i} = \begin{bmatrix} B_{11il_i} \\ B_{12il_i} \end{bmatrix}$, $B_{2i} = \begin{bmatrix} B_{21il_i} \\ B_{22il_i} \end{bmatrix}$, $b_i = \begin{bmatrix} b_{1il_i} \\ b_{2il_i} \end{bmatrix}$, $C_{il_i} = \begin{bmatrix} C_{1il_i} & C_{2il_i} \end{bmatrix}$ and

Any symmetric positive-definite matrix Q_{il_i} can be partitioned into $\begin{bmatrix} Q_{il_i 1} & Q_{il_i 2} \\ Q_{il_i 2}^T & Q_{il_i 3} \end{bmatrix} > 0$, with $Q_{il_i 1} > 0$ and $Q_{jl_j} = \begin{bmatrix} Q_{jl_j 1} & Q_{jl_j 2} \\ Q_{jl_j 2}^T & Q_{jl_j 3} \end{bmatrix} > 0$ are block-partitioned according to the of Q_{il_i} . It is obvious that $Q_{il_i} > 0$ if and only if $\begin{bmatrix} Q_{il_i 1} & 0 \\ 0 & S_{il_i} \end{bmatrix} > 0$. Then, by multiplying Eq.15 with the nonsingular matrix $diag\{\begin{bmatrix} I & 0 \\ -Q_{il_i 2}^T Q_{il_i 1}^{-1} & I \end{bmatrix}, I, I, I, I\}$ to the left and with its transpose to the right. Using the qualities $C_{yil_i} \begin{bmatrix} Q_{il_i 1} & 0 \\ Q_{il_i 2}^T & S_i \end{bmatrix} = Q_{il_i 1} C_{yil_i}$ and the notation $L_{il_i} Q_{il_i 1} = Y_{il_i}$, we obtain exactly inequality (7).

The above discussion leads leads to the following results.

Theorem 1: If there exist symmetric positive definite matrices Q_{1il_i} , Q_{2jl_j} , S_{il_i} ,matrices Y_{il_i} and positive scalars α_i satisfying

$$(16)\quad \min_{Q_{i1} Q_{i12} S_i Y_i Q_{j1} Q_{j2} Q_{j3}} \gamma^2 \text{ subject to LMI (7)and LMI (8)}$$

(i) the discrete-time PWA system (1) is stabilizable by a static output feedback gain $LY_i Q_{i1}^{-1}$ and has the following properties:

(ii)the H_∞ -norm of the corresponding closed-loop system is smaller than $\gamma > 0$,

(iii)the time-domain constrains (2) are respected.

Robust constrained H_∞ control

Based on the results obtained in Section 3, we now handle the time-domain constrain (2) and develop a robust constrained H_∞ controller. The control scheme is firstly geiven as follows.

Theorem 2: Given $r > 0$. Suppose that for all $i, j \in \Omega$, $l = 1,\cdots,L_j$, $l_0 = 1,\cdots,L_{j0}$, where i_0 is the dinex of the subspace that contains $x(0)$, and $w_{\max}$ that is a priori knowledge on disturbance energy bound, the LMI optimisation problem

$$(17)\quad \min_{Q_{i1} Q_{i12} S_i Y_i Q_{j1} Q_{j2} Q_{j3}} \gamma^2$$

subject to LMI (7)and LMI (8)and

$$\begin{bmatrix} r & * \\ y(k) & Q_{il_i1} \end{bmatrix} \geq 0 \tag{18}$$

$$\begin{bmatrix} \dfrac{u_{i,\max}^2}{r} & * & * \\ Y_{il_i}^T Q_{il_i1} & Q_{il_i1} & * \\ E_{1il_i} Q_{il_i1} + E_{2i} Q_{i2}^T & 0 & \alpha_i (f_i f_i^T - 1) \end{bmatrix} > 0 \tag{19}$$

Proof: proper (i) is indicated from the discussion from (7) to (15). In the following, we will prove that the output constrains (2) can be guaranteed by (18) and (19).

In the preceding section, we formulated the SOF problem without input constraints. In this section we show how input constraints can be incorporated as LMI constraints in the robust SOF problem.

Denote

$$\Phi(Q_{11}, r_c) = \left\{ y \in R^m : y(k)^T Q_{11}^{-1} y(k) \leq r_c \right\} \tag{20}$$

By taking the Schur complement, $y(k)^T Q_{11}^{-1} y(k) \leq r$ is equivalent to (18). Hence, LMI (18) forces the output $y(k)$ to be contained in the ellipsoid $\Phi(Q_{11}, r)$. Suppose, now that $y(k) \in \Phi(Q_{11}, r)$ at sampling time k. The constraints (2) are respected

$$\begin{aligned} &\left| u_i(k) \right|^2 \leq \max_{y \in \Omega(Q_{11,k}, r)} \left| (YQ_{11}^{-1})_i\, y(k) \right|^2 \leq \\ &r \left\| (YQ_{11}^{-\frac{1}{2}})_i \right\|_2^2 = (YQ_{11}^{-1} Y^T)_{ii} \leq u_{i,\max}^2 \end{aligned} \tag{21}$$

we obtain (19) guarantees that $\left| u_i(k) \right| \leq u_{i,\max}$, $i = 1, \cdots, m$.

Conclusions

In the framework of LMI optimisation, this paper designs a H_∞ static output feedback controller for constrained discrete-time PWA systems. The designed controller has a certain disturbance attenuation level, and can guarantee the satisfaction of constraints with the disturbance energy.

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REFERENCES

[1] Gao yahui., Liu zhiyuan and Chen hong. Robust H∞ Control for Constrained Discrete- Time Piecewise Affine Systems with Time-Varying Parametric Uncertainties, *IET Control Theory and Applications*., 8(2009), 1132-1144

[2] Cairano S. D, Bemporad A. An equivalence result between linear hybrid automata and piecewise affine systems, *Proceedings of the 45th IEEE Conference on Decision and Control CA*, USA, *San Diego.*, (2006), 2631-2636

[3] Li chuandong, Liao xiaofeng and Yang xiaofan. Swich control for piecewise affine chaotic systems, *Chaos*., (2006)16, 033104

[4] Rodrigues L, Hassibi A and J. P How. Illinois., Output feedback controller synthesis for piecewise-affine systems with multiple equilibria. *Proceedings of the American Control Conference, Chicago*, (2000), 1784-1789

[5] Boyd S., Ghaoui L., Feron E and Balakrishnan V, Linear Matrix Inequalities in System and Control Theory. Philadephia, (1994)

[6] Johansson M., Rantzer A., Computation of piecewise quadratic Lyapunov functions for hybrid systems, *IEEE Trans. Autom. Control*, 43(1998), No. 4, 555–559

[7] Hassibi A., Boyd S., Quadratic stabilization and control of piecewise-linear systems, *Proc. American Control Conf., Philadelphia, Pennsylvania*, June (1998), 3659–3664

[8] Mingnone D., Trecate G.F., Morari M., Stability and stabilization of piecewise affine and hybrid systems: an LMI approach, *Proc. 39th IEEE Conf. Decision and Control, Sydney, Australia, December* , (2000), 504–509

[9] Cuzzola F.A., Morari M., A generalized approach for analysis and control of discrete– time piecewise affine and hybrid systems, *Hybrid systems: computation and control lecture notes in computer sciences*, 2034 (2001), 189–203

[10] Trecate G.F., Cuzzolz F.A., Mingnone D., Morari M., Analysis and control with performance of piecewise affine and hybrid systems, *Proc. American Control Conf., Arlington, VA, June* ,(2001), 200–205

[11] Trecate G.F., Cuzzola F.A., Mingnone D., Morari M., Analysis of discrete-time piecewise affine and hybrid systems, *Automatica*, 38(2002) , No.12, 2139–2146

[12] Feng G., Stability analysis of piecewise discrete-time linear systems, *IEEE Trans. Autom. Contro*l, 47(2002),No.7, 1108–1112

[13] Morinaga E., Hirata K., An L_2-gain analysis of piecewise affine systems by piecewise quadratic storage functions, *Proc. American Control Conf., Boston, Massachusetts,June*,(2004), 5176–5181

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