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Linear algebra approach and the quasi-Newton algorithm for the optimal coil design problem

Abstract. *In this work, the problem of shaping magnetic field excited by magnetic coils is considered. An important special case is constructing coils that excite a homogenous magnetic field in some specific regions. This problem is an important step in the design of superconducting magnetic coils for Magnetic Resonance Imaging devices, where in certain regions a strong static magnetic field with high homogeneity is needed. The linear coil design problem is investigated using two approaches. The first approach is based on linear algebra. The problem under study can be formulated as an over-determined set of linear equations. Since usually the matrix describing the problem is ill-conditioned, solutions obtained using the least squares method are very large in magnitude and hence are useless from the applications point of view. The Tichonov regularization method is employed to make the linear problem well-posed. In the second approach the problem is formulated as an optimisation task, which is solved using the quasi-Newton optimisation algorithm. Performance of both methods in terms of their effectiveness is compared using several examples.*

Streszczenie. *W niniejszej pracy autor rozważa problem kształtowania pola magnetycznego w cewce. Ważnym szczególnym przypadkiem jest konstrukcja cewek wytwarzających pole jednorodne w zadanym obszarze. Ten problem szczególnie dotyczy budowy cewek dla urządzeń rezonansu magnetycznego, gdzie w pewnym rejonie wymagane jest jednorodne pole magnetyczne. Rozważane są dwie metody rozwiązania tego problemu. Pierwsza z nich korzysta z metod algebry liniowej, na potrzeby których problem formułuje się jako nadokreślony układ równań liniowych. Problem ten zwykle jest źle uwarunkowany, szczególnie dla wysokiego wymiaru problemu. W efekcie rozwiązania otrzymane za pomocą metody najmniejszych kwadratów mają bardzo duże wartości bezwzględne i są nieprzydatne punku widzenia aplikacji. W celu poprawy wskaźnika uwarunkowania problemu zastosowano metodę regularyzacji Tichonowa. W drugim podejściu problem jest sformułowany jako zadanie optymalizacyjne, do którego rozwiązania zastosowano metodę quasi-Newtona. Oba podejścia zostały porównane, ze względu na efektywność oraz możliwość aplikacji w układach rzeczywistych.* (***Zastosowanie liniowej algebry i algorytmu quasi Newtona do optymalnego projektowania cewki)***

Keywords: coil design, the least squares method, Tichonov regularization, the quasi-Newton method
Słowa kluczowe: projektowanie cewek, metoda najmniejszych kwadratów, regularyzacja Tichonowa, metoda quasi-Newtona

Introduction

One of the most important problems in electromagnetic devices design is to obtain a required magnetic or electric field distribution in a given region. Frequently, this problem is solved using stochastic algorithms, like simulated annealing, genetic algorithms and evolutionary computations [1, 2, 3]. However, this is not always the best solution and choosing the best optimiser to a concrete problem is an important issue. In [4], it was shown that in many cases deterministic algorithms are more efficient and takes less computational time than stochastic algorithms. In this work, the comparison of two deterministic approaches to the problem of optimal coil design is performed.

Problem description

The problem studied in this work is to design a coil producing a required magnetic field in a given region. This problem can be solved by using the idea of the "target field" approach applied to the problem of static magnetic field excited by the set of n coaxial coils [5].

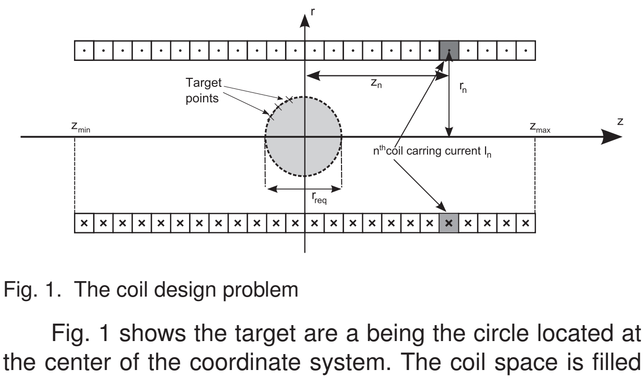

Fig. 1. The coil design problem

Fig. 1 shows the target are a being the circle located at the center of the coordinate system. The coil space is filled by an array of individual coils. The coils are assumed to be ideal current loops located at the center of the little squares. The goal is to find the values of currents flowing through coils that excite the desired magnetic field at each target point. In [6] can be noticed that analitical approach to the problem of magnetic field comutation in MRI devices is quite complicated and it is usefull to use the simplification applaied here.

Table 1. Dimensions of the coil and target areas

Coil dimensions	
r_{coil}	5 cm
z_{coil}	20 cm
Target area, case 1	
r_{req}	4 cm
Target area, case 2	
z_{req}	30 cm

We consider the problem of designing a coil of length z_{coil} and radius r_{coil}. Two examples of target areas are considered. In the first case target points are distributed on the circle of radius r_{req} positioned at the centre of coordinate system. In the second case target points are located on the interval of length z_{req} enclosed in the z axis and centered at the origin. The parameters are presented in the Table 1.

For the system of coaxial circular coils, only the B_z and B_r components of the magnetic field in the z and r directions need to be considered. In [7], it was shown that B_r component is much smaller than B_z and it has negligible contribution on the total magnetic field. This effect is often called the quadrature suppression. Therefore, it is sufficient to consider the B_z component only. Each coil generates a magnetic field contribution at each target point. A contribution from a single coil with the current i_k located at (r_k, z_k) to the magnetic field at the target point (r_j, z_j) can be calculated using the following relation:

$$(1)\quad B_{k,j} = \frac{\mu_0 i_i}{2\pi\sqrt{(r_k + r_j)^2 + (z_j - z_k)^2}} \cdot \left(K(k) - \frac{r_k^2 + r_j^2 + (z_j - z_k)^2}{(r_k - r_j)^2 + (z_k - z_j)^2} E(k) \right).$$

where

$$k = \sqrt{\frac{4 r_k r_j}{(r_k + r_j)^2 + (z_k - z_j)^2}},$$

and $K(\cdot)$, $E(\cdot)$ denote the elliptic integrals of the first and second kind, respectively. When the target points are located on the z axis ($r_j = 0$), the equation (1) reduces to:

(2) $$B_{k,j} = \frac{\mu_0 i_k r_k^2}{2(r_k^2 + (z_k - z_j)^2)^{\frac{3}{2}}}.$$

Note that formulas (1) and (2) are linear with respect to currents i_k.

The linear algebra approach

Let us assume that there are n individual coils with currents $i_1, i_2, \ldots, i_n$ and $m > n$ target points with the desired value of the magnetic field $b_1, b_2, \ldots, b_m$. Since the relation between the field at target points and the current at a given coil is linear, one can formulate the problem as an overdetermined set of linear equations:

(3) $$Ai = b$$

where $A \in \mathbb{R}^{m,n}$ is the coefficient matrix, $i \in \mathbb{R}^n$ is the vector of the currents to be found, and $b \in \mathbb{R}^m$ is the vector describing the required field at the target points. Components of the matrix A can be computed by dividing the result obtained from the relation (1) or (2) by the current i_k. The least-squares solution is the one that minimizes the sum of squares of residual errors for all target points. This can be expressed as

(4) $$\min_i \left\{ ||Ai - b||_2^2 \right\}.$$

It is well known that the minimum (4) can be found by solving the set of normal equations

(5) $$A^T A\, i = A^T b.$$

The design goal is to achieve the homogeneous magnetic field in the target region (i.e. $b_k = \mathrm{const}$). Fig. 2 presents the least square solution for the case $n = 6$, $m = 50$. Note that the currents in neighboring coils have opposite signs. This is undesirable from the applications point of view. Fig. 3 presents the solution for the case $n = 20$. Like in the previous case the currents have opposite signes. Another important observation is that the values of currents are very large (some current are above $4000\mathrm{A}$). From the practical point of view this solution is useless.

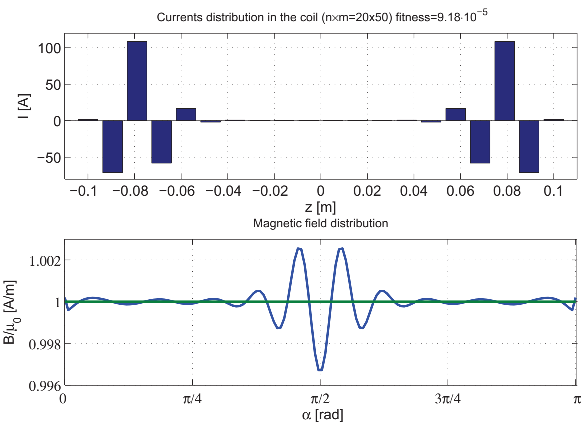

Fig. 2. Solution and the corresponding field distribution, for the target points located on the circle placed at the center of coordinate system.

In order to better understand the problem considered, let us now compute its condition number. The condition number of the square matrix C is defined as:

(6) $$\kappa(C) = \|C^{-1}\| \cdot \|C\|,$$

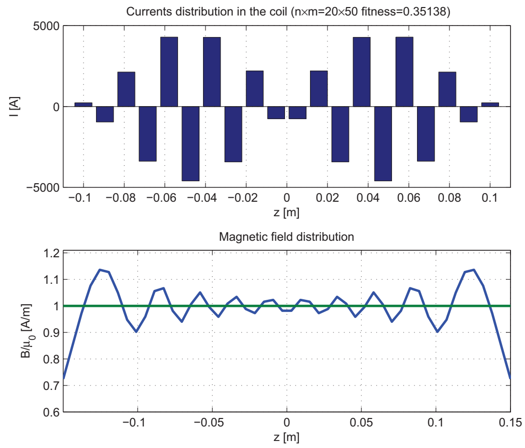

Fig. 3. Solution and the corresponding field distribution, for the target points located on the z axis.

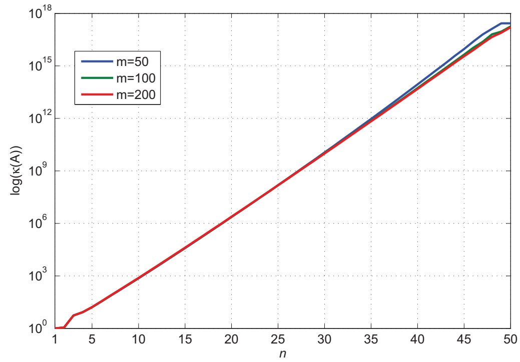

Fig. 4. Condition number κ of the matrix $A \in \mathbb{R}^{n \times m}$ in the function of dimensions m and m

where $\| \cdot \|$ is a matrix norm. In the following, the matrix norm induced by the Euclidean norm is used. Unfortunatelly because of non-square matrix A (the problem is overdetermined) the formula (6) can not be applied. The condtition number here need to be understood as a meaure of rank deficiency, and is deffined as [8]:

(7) $$\kappa(C) = \frac{\sigma_{max}(A)}{\sigma_{min}(A)}$$

Where $\sigma_{max}(A)$ and $\sigma_{min}(A)$ are the maximum and minimum singular values, respectively. Fig. 4 presents the condition number defined as (7) of the matrix A as a function of n. The condition number is growing rapidly with the size of the problem, and for $n \geq 25$ it becomes larger than 10^8. It is clear that for large n small changes in the vector b may cause large variations in the solution i. The results presented above show that the problem for large n is ill-conditioned.

Tichonov regularization

To improve the smoothness of the solution the Tichonov regularization [9] is used. To this end additional information (the so-called a priori information) is imposed. Since we require that the solution is smooth, the criterion (4) can be reformulated as

(8) $$\min_i \left\{ ||Ai - b||_2^2 + \lambda^2 ||i||_2^2 \right\}$$

The regularization parameter λ controls the smoothness of the solution. The solution of the problem (8) has the form

(9) $$i = (A^T A + \lambda I_n)^{-1} A^T b,$$

where $I_n \in \mathbb{R}^n$ is the identity matrix.

The problem is to find the value of the parameter λ that provides a good compromise between the desired field shape and the smoothness of the solution. There are many techniques which estimate the optimal value. One of the popular methods is the L-curve method [10]. The L-curve is a parametrized (λ is a parameter) plot of the norm of the solution $||i||_2$, versus the residual norm $||Ai - b||_2$. An important feature of the L-curve is that its corner appears for a regularization parameter close to the optimal value. The idea of the L-curve criterion for finding the optimum value of the regularization parameter λ is to choose a point on this curve as close as possible to the corner. Unfortunately, for our purpose the above method is not working properly. In out case, the L-curve is lacking the L-shape, and is hard to estimate its corner. Instead, we choose minimum $\lambda > 0$ such that all elements of the solution (9) are non-negative, i.e.

(10) $$\lambda_{\text{opt}} = \min\{\lambda\colon i_k \geq 0 \text{ for all } k\}.$$

Fig. 5 presents the solution of (8) obtained for the optimal value of the regularization parameter λ_{opt}=$9.5502{\cdot}10^{-2}$. It can be seen that the Tichonov regularization smoothes the distribution of currents. Note that all currents have the same sign, and their values are much smaller than for the least squares solution (see Fig. 3). All calculations were made using the Matlab environment and the regularization toolbox [11].

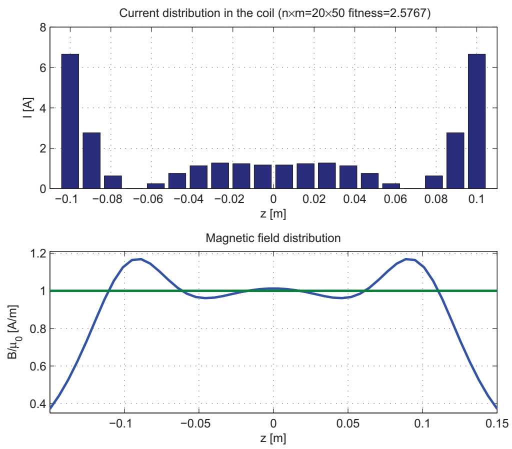

Fig. 5. Regularized solution for optimal parameter λ and the corresponding field distribution, for the target points located on the z axis.

The results for the case when the target points are located on the circle are presented in Fig. 6. One can see that the solution is smooth and the values of currents are acceptable, especially when compared to the solution presented in Fig. 2.

The quasi-Newton method

The quasi-Newton method is a deterministic optimisation method [12]. This method belongs to the wide class of the iteration gradient methods. At each iteration step $k \in \{1, 2, \ldots\}$, the search direction d^k is determined using:

$$d^{(k)} = -H_k^{-1} \nabla f(x^{(k)})$$

where H_k is a symmetric positive define matrix that approximates the hessian matrix $\nabla^2 f(x^{(k)})$ of f at $x^{(k)}$. An approximation of the inverse hessian matrix $B_k = H_k^{-1}$ is computed using Broyden-Fletcher-Goldfarb-Shanno (BFGS) formula:

$$B_{k+1} = B_k + \frac{y^{(k)} y^{(k)T}}{y^{(k)T} \Delta x^{(k)}} - \frac{B_k \Delta x^{(k)} (B_k \Delta x^{(k)})^T}{\Delta x^{(k)T} B_k \Delta x^{(k)}}$$

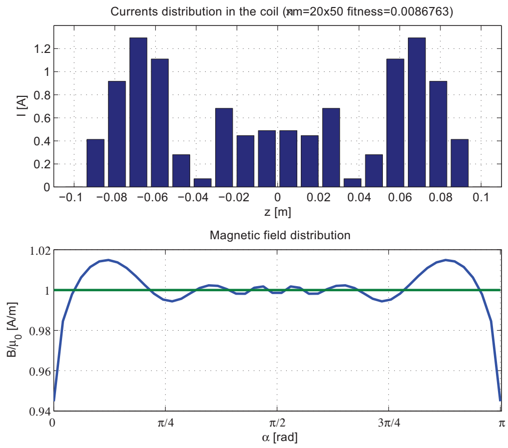

Fig. 6. Regularized solution and the corresponding field distribution, for the target points located on the circle placed at the center of coordinate system.

Usually, the identity matrix $B_0 = I$ is selected as a starting point. The quasi-Newton iteration formula is used in the search process:

(11) $$x^{(k+1)} = x^{(k)} - B_k \nabla f(x^{(k)})$$

The following objective function is used.

(12) $$f(x) = ||b - Ax||^2.$$

In order to compute the next iteration using formula (11) one has to calculate the gradient of the objective function $\nabla f(x^{(k)})$. In our case the objective function (12) is quadratic and has one optimal point (global solution). But when the dimension of the problem is large, the function becomes very flat, and it is hard to compute the gradient of f numerically. Instead of calculating the derivatives numerically the exact formula for the gradient is used (13). This approach significantly improves the speed and the efficiency of the algorithm.

(13) $$f'(x) = -2A^T b + 2A^T Ax.$$

Quasi Newton solutions

The solutions for the unconstrained optimisation problem found using the quasi-Newton method are exactly the same as the solutions obtained with the least squares method. In order o force the currents to be positive the objective function (12) is modified by adding a penalty factor:

(14) $$f_{pen}(x) = f(x) + \beta \sum_{i=1}^{n} (|x_i| - x_i).$$

$\beta \in \mathbb{R}$ is the parameter controlling the influence of the penalty factor. The large value of $\beta = 100$ is used to make sure that all currents are positive.

Fig. 7 presents the solution obtained with the quasi-Newton algorithm for the case when the target points are located on the z axis. Let us note that solutions found using the quasi-Newton algorithm depend on the starting point x_0. Approximately in each fifth run a solution similar to the one presented in Fig. 7 is found. Observe that this solution is better (the value of the objective function is 20% less) than the one found using the Tichonov regularization.

Fig. 8 presents the quasi-Newton solution for the case when the target points are distributed on the circle located in the centre of the coordinate system. Here it can be easily seen that the currents distribution lacks the symmetry. This is

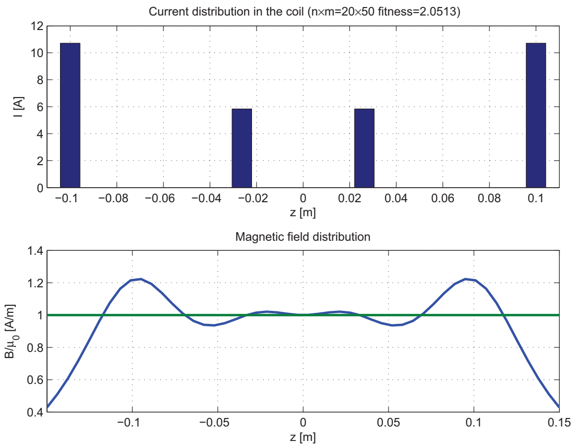

Fig. 7. quasi-Newton solution and the corresponding field distribution, for the target points located on the z axis

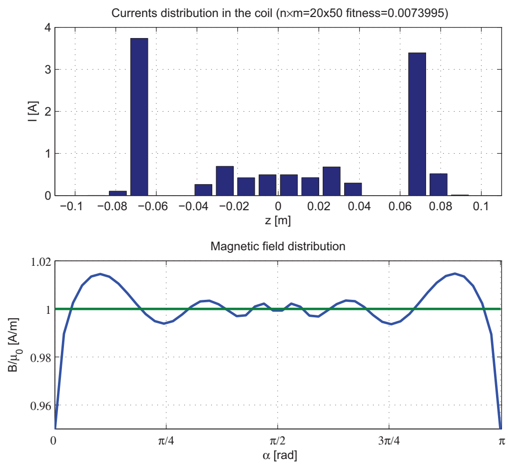

Fig. 8. quasi-Newton solution and the corresponding field distribution, for the target points located on the circle placed in the center of the coordinate system

due to the fact that together with the problem dimensions, the quadratic function is becoming very flat. At certain iterations the values of the gradient become very small and the algorithm cannot find a proper direction for the minimisation of the objective function.

Unlike the previous figures Fig. 9 presents the quasi-Newton solution for the case when the currents are distributed on 2D plane. Here the plane was devided by 6×6 simple coils.

Conclusion

In this work, two deterministic methods for solving a coil design problem was compared. For the unconstrained problem the least square solution is easy and finds proper solutions in the reasonable time. But from the applications point of view the solutions migh be useless (high values of currents, opposite currents directions in neighboring coils). For higher dimensional problems the matrix becomes ill-conditioned.

When the constrained problem is considered, the Tichonov regularization is very useful. It smoothes the problem and in this process some of the design constrains are fulfilled. Values of currents have smaller amplitudes and all currents are in the same directions.

Using the quasi-Newton method one can find better solutions then with the Tichonov regularization. A disadvantage is that it is necessary to run the algorithm from many different starting points in order to make sure that the best solution is found. From the application point of view the lack of symmetry can be fixed during the real coil design process. The quasi-Newton solutions give a good hint what should be the distribution of currents to obtain the optimal coil.

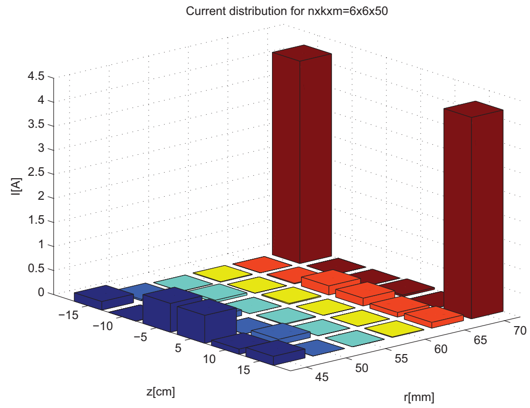

Fig. 9. Current distribution for situation when currents are distributed on 2D plane devided by 6x6.

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