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Characterization of the shape of unknown objects by inverse numerical methods

***Abstract**. The problem of the image reconstruction in Electrical Impedance Tomography (EIT) is a highly ill-posed inverse problem. There are mainly two categories of image reconstruction algorithms, the direct algorithm and the iterative algorithm which was used in this publication. The forward problem can be solved by the finite element method, immersed interface method or boundary element method. The representation of the shape of the boundary and its evolution during an iterative reconstruction process is achieved by the level set function, the Chan-Vese model or by the variational level set method.*

***Streszczenie**. W pracy przedstawiono metodę rozwiązania zagadnienia odwrotnego w tomografii impedancyjnej opartą na idei zbiorów poziomicowych oraz modelu Mumforda-Shaha. Algorytmy numeryczne rozwiązania są odpowiednią kombinacją wymienionych metod oraz metody elementów skończonych, za pomocą której wyznaczana jest konduktywność poszukiwanych obiektów. (**Praktyczne implementacje metody zbiorów poziomicowych w tomografii impedancyjnej**).*

Keywords: Level Set Method, Mumford-Shah model, Electrical Impedance Tomography, Finite Element Method
Słowa kluczowe: metoda zbiorów poziomicowych, tomografia impedancyjna, metoda elementów skończonych

Introduction

There was discussed the application of the level set function for identifying the unknown shape of an interface in a problem motivated by electrical impedance tomography. Level set methods, variational level set methods and the Chan-Vese model were chosen for electrical impedance tomography. It is sometimes more important to recover the shape of the domains containing different materials than to recover the values for the materials. The level set function techniques was shown to be successful to identify the unknown boundary shapes. In the model problem from electrical impedance tomography is required to identify unknown conductivities from near-boundary measurements of the potential [1,7,9].

Topological Methods

Level Set Methods

The level set method tracks the motion of an interface by embedding the interface as the zero level set of the signed distance function. The motion of the interface is matched with the zero level set, and the resulting initial value partial differential equation for the evolution of the level set function. The idea is merely to define a smooth function $\phi(x,t)$, that represents the interface as the set where $\phi(x,t)=0$. The motion is analyzed by the convection the $\boldsymbol{\phi}$ values (levels) with the velocity field. The Hamilton-Jacobi equation of the form [4,5,8]

$$\frac{\partial\phi}{\partial t}+\mathbf{v}\cdot\nabla\phi=0 \tag{1}$$

where **v** is the velocity on the interface.

When flat or steep regions complicate the determination of the contour, the reinitialization is necessary. This reinitialization procedure is based by replacing by another function that has the same zero level set but behaves better. This is based on following partial differential equation:

$$\frac{\partial}{\partial t}\phi+S(\phi)(\nabla\phi-1)=0 \tag{2}$$

where S($\boldsymbol{\phi}$) is defined as:

$$S(\phi)=\begin{cases}-1 & \text{for } \phi<0\\ 0 & \text{for } \phi=0\\ 1 & \text{for } \phi>0\end{cases} \tag{3}$$

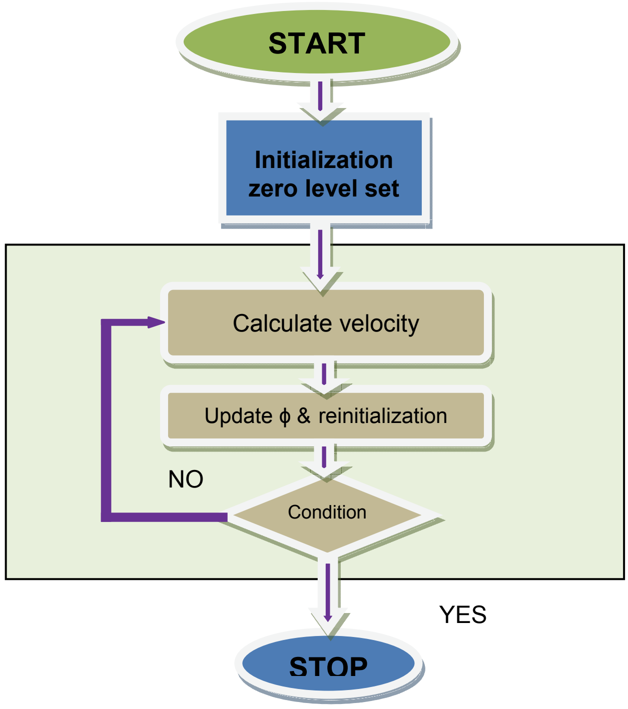

Fig. 1. The scheme of the algorithm – the level set method

Variational Level Set Methods

The formulation of the variational level set method consists of an internal energy term that penalizes the deviation of the level set function and an external energy term that drives the motion of the zero level set toward the desired image features. When flat or steep regions complicate the determination of the contour, the reinitialization is necessary. This reinitialization procedure is based by replacing by another function that has the same zero level set but behaves better. Variational formulation for geometric active contours that forces the level set function to be close to a signed distance function, and therefore

completely eliminates the need of the costly reinitialization procedure.
The resulting evolution of the level set function is the gradient flow that minimizes the overall energy functional [2,6]:

(4) $$\boldsymbol{P}(\phi)=\int_{\Omega}\frac{1}{2}(|\nabla\phi|-1)^2\,\mathrm{d}x\mathrm{d}y$$

An external energy for a function **ϕ**(x, y) is defined as below:

(5) $$\boldsymbol{E}(\phi)=\mu\,\boldsymbol{P}(\phi)+\boldsymbol{E}_m(\phi)$$

where

$\boldsymbol{P}(\phi)$ – internal energy, $\boldsymbol{E}_m(\phi)$ – external energy.

Denoting by $\frac{\partial \mathbf{E}}{\partial \phi}$ the Gateaux derivative of the functional **E** receiving the following evolution equation:

(6) $$\frac{\partial\phi}{\partial t}=-\frac{\partial \mathbf{E}}{\partial\phi}$$

The level set function can be written as:

(7) $$\frac{\partial\phi}{\partial t}=\mu\left[\Delta\phi-\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right]+\nu\delta(\phi)$$

where **δ** is the delta function.

Chan-Vese Model

For more than two phases was introduced the multiple level sets idea by Vese and Chan [10]. The algorithm set formulation for the general minimization problem in image processing, to compute piecewise-smooth optimal approximations of a given image. The proposed model follows and fully generalizes works [3,10], where there was proposed an active contour model without edges based on a 2-phase segmentation and level sets. The piecewise-constant segmentation of the image allows for more two segments using a new multi-phase level set formulation and partition of the image domain. For more than two phases was introduced the multiple level sets idea by Vese and Chan. The algorithm sets a formulation and models for the general Mumford-Shah minimization problem in image processing, to compute piecewise-smooth optimal approximations of a given image. The problem can be easily generalized to the case where the domain contains more than two materials.

(8) $$F(s,C)=\omega\,\boldsymbol{L}(C)+\eta\int_{\Omega}(\mathrm{s_o}-\mathrm{s})^2\mathrm{d}\Omega+\int_{\Omega\setminus C}|\nabla \mathrm{s}|^2\mathrm{d}\Omega$$

There was proposed an active contour model without edges based on a 2-phase segmentation and level sets γ_{I} and γ_{II}. Conductivity **γ** is represented as:

(9) $$\gamma=\gamma_{\mathrm{I}}\mathbf{H}(\phi)+\gamma_{\mathrm{II}}(1-\mathbf{H}(\phi))$$

where **H** is the Heaviside function.
The derivative of F with respect to **γ** is given by

(10) $$\left[\frac{\partial \mathrm{F}}{\partial\gamma}\right]=-\sum_{j=1}^{\mathrm{p}}\nabla\mathbf{u}_j\nabla\mathbf{p}_j$$

where: **u** – the electric potential, **p** – the adjoint variable.
Level set function is updated the following iterative scheme:

(11) $$\phi^{k+1}=\phi^k-\mu\left[\frac{\partial \mathrm{F}}{\partial\phi}\right]$$

where coefficient μ>0 and

(12) $$\left[\frac{\partial \mathrm{F}}{\partial\phi}\right]=\left[\frac{\partial \mathrm{F}}{\partial\gamma}\right]\frac{\partial\gamma}{\partial\phi}=\left[\frac{\partial \mathrm{F}}{\partial\gamma}\right](\gamma_{\mathrm{I}}-\gamma_{\mathrm{II}})\,\boldsymbol{\delta}(\phi)$$

where δ is the Dirac delta function.

Variational Chan-Vese Model

The formulation of the variational Chan-Vese model consists with the variational level set method and the Chan-Vese algorithm.
The process for minimization of the functional is the following:

(13)
$$\phi_t^{k+1}=\phi_t^k-\Delta t\left[\begin{array}{l}\mu_1\left(\Delta\phi-\mathrm{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right)\right)\\+\mu_2\left(\nabla\lambda_{\mathrm{k}}\cdot\nabla\varphi_{\mathrm{k}}\right)(\gamma_{\mathrm{I}}-\gamma_{\mathrm{II}})\,\delta(\phi)\end{array}\right]$$

where μ_1, μ_2 are coefficients.

Numerical algorithms in Electrical Impedance Tomography

Electrical impedance tomography is a widely investigated problem with many applications in physical and biological sciences. It is well known that the inverse problem is nonlinear and highly ill-posed. The forward problem in EIT is solving by Laplace's equation. The objection function is minimized (the difference between the potential due to the applied current and the measured potential).

The figure 2 presents the process reconstruction in EIT. The numerical model was inserted in the inside of the examined object. The representation of the shape of the boundary and its evolution during an iterative reconstruction process is achieved by the numerical algorithms. The images show the original object and reconstruction after the process iterations. In the examples were used the zero level set function as an initial condition representing by the different shapes. The final contour represents the zero value of the level set function. The reconstruction is correct, because the region borders nearly are located the object edges. The numerical model was inserted in the inside of the examined object. The grid was used by 32x32 elements solution, the number of these variables are equal to 1089. The pictures show different objects and the process reconstruction. The original object is noted by the blue line, zero level set function is red, the following iterations are green and the final figure is pink. The images show the original object and reconstruction after the process iterations. In the example was used the zero level set function as an initial condition representing by the circle. The final contour represents the zero value of the level set function. The reconstruction is good, because the region borders nearly are located the object edges. Using different the zero level set function, the object function achieves the minimum after the various number of iterations.

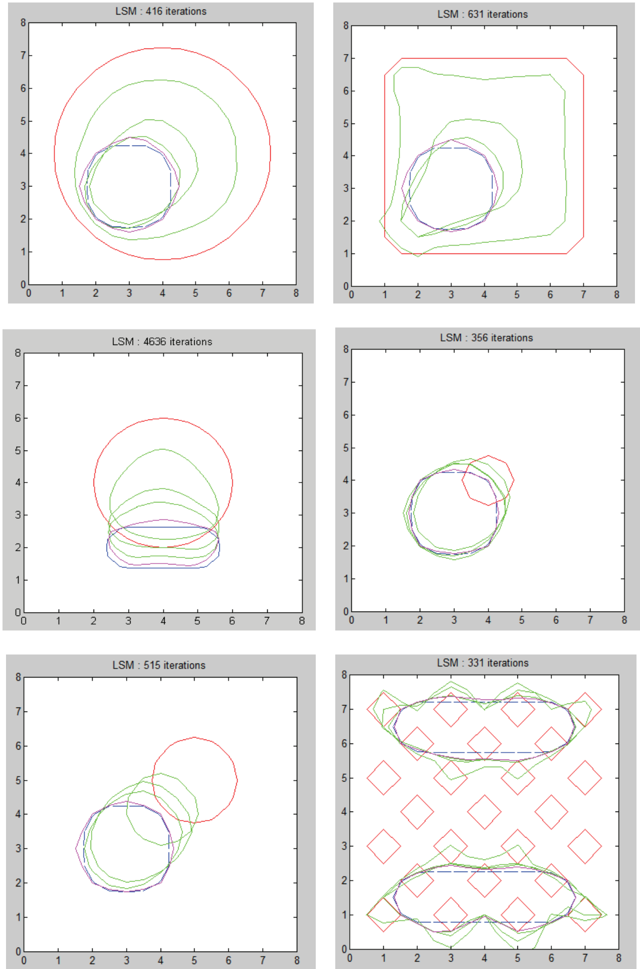

Fig.2 The image reconstruction with different zero level set functions

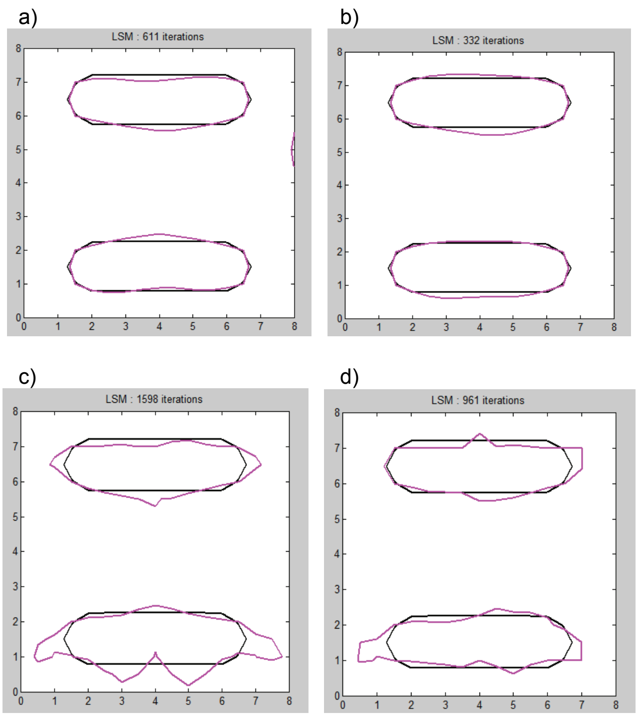

Fig.3 Comparing methods – 2 objects: a) the level set method, b) Chan-Vese model, c) the variational level set method, d) the variational Chan-Vese model

Topological methods of the image reconstruction were shown on the figure 3. Four methods (2 objects) were compared on these pictures: the level set method, Chan-Vese model, the variational level set method, the variational Chan-Vese model.

Summary

This paper has introduced the new topological method to solve the inverse problem of approximation of material coefficient. The applications of the level set function, the variational level set method, Chan-Vese model for the electrical impedance tomography was presented in this paper. The level set function techniques was shown to be successful to identify the unknown boundary shapes. Variational formulation is the faster the traditional level set because completely eliminates the need of the costly reinitialization procedure. The Chan-Vese model gives the best quality reconstruction unknown areas with many objects.

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