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Fully Adaptive Higher-Order Finite Element Model of Electric Field near High-Voltage Insulator for Outdoor Use

***Abstract**. Problems of automatic adaptivity in finite element method of higher order of accuracy are discussed. Particular attention is paid to the hp-adaptivity that exhibits the highest level of flexibility and extremely fast convergence. The theoretical aspects are illustrated by an example of magnetic field near a high voltage insulator for outdoor use solved numerically by our own code Agros2D. The results are compared with data obtained by other commercial codes*

***Streszczenie.** Przedyskutowano problem automatycznej adaptacji metody elementów skończonych. Szczególną uuwagę zwrócono na adaptacyjność typu hp charakteryzującą się dobrą elastycznością i szybką zbieżnością. Zagadnienie rozważano na przykładzie analizy pola magnetycznego w pobliżu izolatora wysokonapięciowego. (**W pełni adaptacyjna metoda elementów skończonych na przykładzie analizy pola magnetycznego w pobliżu izolatora wysokonapięciowego**)*

Keywords: automatic adaptivity, higher order finite element method, electric field, outdoor insulator.
Słowa kluczowe: metoda elementów skończonych, adaptacyjność**.**

Introduction

The automatic adaptivity represents the integral part of all top-level computer codes. It serves for reduction of errors brought about by the numerical solution of the given problem. The corresponding algorithms are applied at the moment when the local error of solution is higher than the acceptable tolerance. This error defined as the difference between the current numerical solution and exact solution may be caused by locally rougher mesh, presence of one or more singular points, curvilinear boundaries or interfaces approximated by polygonal lines, etc. Such errors must be identified in the course of computation and appropriate measures have to be taken for their reduction.

The adaptive techniques are also implemented in the codes Agros2D [1] and Hermes [2] that have been developed for a couple of years in our group. Agros is a powerful user's interface intended for pre-processing and post-processing of the problem, while Hermes is a library of advanced numerical procedures for monolithic and fully adaptive solution of systems of generally nonlinear and nonstationary partial differential equations (PDEs) based on the finite element method of higher order of accuracy. Both codes written in C++ and are used for solving complex coupled problems rooting in various domains of physics. They are freely distributable under the GNU General Public License.

Elements of adaptivity in codes Agros and Hermes

The algorithms of automatic adaptivity implemented in Agros2D and Hermes are divided into the following principal groups:

- Refinement of elements in regions where the solution exhibits an unacceptable error (*h*-adaptivity). While the original large finite element is split to several smaller elements, the degree of the polynomials replacing the real distribution of the investigated quantity in them remains the same.
- Improvement of approximation of the investigated quantity (*p*-adaptivity). Now the shapes of elements in the region do not change, but we increase the orders of the polynomial approximating the distribution of the investigated quantity.
- The combination of both above ways (*hp*-adaptivity). This belongs to the most flexible and powerful techniques characterized by the exponential convergence of results.
- Work with an arbitrary combination of triangular, quadrilateral and curved elements. Particularly using the curved elements for modelling the curved boundaries or interfaces leads (in comparison with classical triangular elements) to considerable savings in the degrees of freedom (DOFs) while the accuracy of results remains the same or is even better.

The *h*-adaptivity is the simplest one and is implemented in numerous existing codes. The elements burdened by high errors are divided into several smaller elements, but there exist even other possibilities. The principal problems accompanying this type of adaptivity are the hanging nodes appearing on the interfaces between the refined elements and elements without refinements. These nodes must be handled with particular care, otherwise they may significantly contribute to the growth of the degrees of freedom of the problem solved [3].

The *p*-adaptivity is even simpler to implement, because the mesh does not change. Only in the selected elements we enlarge the order of the corresponding approximating polynomials.

The *hp*-adaptivity represents the most complicated method and its implementation is highly nontrivial. On the other hand, it exhibits the exponential convergence of results and was proven to be an extremely powerful tool just in the finite element methods of higher orders of accuracy. Although we can meet this method still very scarcely, its algorithms are implemented both in Agros2D and Hermes.

Errors and selection of adaptive technique

As told before, algorithms of adaptivity start to be applied at the moment when some local error of solution is higher than the acceptable tolerance. Consider first an equation

(1) $$Lf = 0$$

where L denotes a differential operator and f a function whose distribution over some domain Ω is to be found. If $f\,'$ is its approximation obtained by the numerical solution of (1), the absolute and percentage relative errors δ and η are defined by the relations

(2) $$\delta = f - f\,' ,\quad \eta = 100\left|\delta / f\right| .$$

Other quantities, that can be used for the evaluation of solution, are norms. Hermes2D works with the classical energetic norm, L^2 norm and H^1 norm [4]

Unfortunately, the exact solution f is known only in very simple analytically solvable cases. Moreover, even when for various (mostly linear) classes of PDEs there exist methods of estimation of the error of solution, we have no tools for estimating it in case of a general nonlinear PDE. That is why we work [4] with the reference solution f_{ref} instead, that is obtained either by a refinement of the whole mesh (*h*-adaptivity), by enlargement of the polynomial degree (*p*-adaptivity) or by both above techniques (*hp*-adaptivity). In this manner we get the candidates for adaptivity even without knowledge of the exact solution f. And this way does not depend on the type of equation to be solved.

Before the adaptivity loop is applied, the code must initialize the refinement selector that determines how the elements should be refined. The selector performs the following steps:

- selection of the candidates for refinement,
- computation of their local errors, which is realized by projecting the reference solution on their FE spaces,
- computation of the number of the degrees of freedom for every candidate,
- evaluation of the score for each candidate and sorting the candidates according to their values,
- selection of the candidate with the highest score.

As mentioned before, the adaptivity algorithm in Hermes needs an actual mesh solution and another solution realized on globally refined mesh (the reference solution). These solutions are subtracted in each adaptivity step in order to obtain an error estimate (as a function of the position). This function is used to decide which elements need to be refined and in which way. Hence, the adaptivity loop begins with the global refinement of the mesh and calculation of the reference solution.

If the error is higher than a given threshold, the adaptation process is started. The calculated local error in the candidate is first weighted with respect to the way of adaptivity that should be used. This weight w is selected in the following way:

- $w = 2$ for the *h*-adaptivity,
- $w = 1$ for the *p*-adaptivity,
- $w = \sqrt{2}$ for the *hp*-adaptivity.

The score s of a candidate is given by the formula

$$s = \frac{\log_{10}\left(\dfrac{\delta}{\delta_0}\right)}{\left(d - d_0\right)^{\xi}}, \qquad (3)$$

where δ is the estimated error in the candidate, d denotes its number of DOFs in the candidate, δ_0 and d_0 are selected parameters and ξ stands for the convergence exponent.

Illustrative example

The selected example is aimed at mapping of the distribution of electric field near a high-voltage insulator for outdoor use. The insulator is made of a ceramic material whose relative permittivity $\varepsilon_{\mathrm{r}} = 6$. Knowledge of the electric field distribution along the surface of the insulator is crucial for its design. Generally used threshold value for its electric field is 450 kV/m.

The basic geometrical dimensions of the insulator under investigation are given in the left part of Fig. 1; its right part shows the boundary conditions. The voltage of the source electrode is $U = 36$ kV and the lower part of the insulator is grounded. The arrangement is considered axi-symmetric and the insulator is surrounded by air.

The governing equation describing the distribution of electric potential in the system reads

$$\operatorname{div}\left(\varepsilon \operatorname{grad}\varphi\right) = 0\,. \qquad (4)$$

The boundary condition along the artificial boundary is of the Neumann's type.

The right part of Fig. 2 depicts the final appearance of the complete mesh after the adaptive process (the numbers in the rectangles showing the orders of the corresponding polynomials, the interface insulator–air being modelled by the curved elements), the right part the distribution of electric potential. Figure 3 then shows the zoom of electric field in the upper part of the insulator.

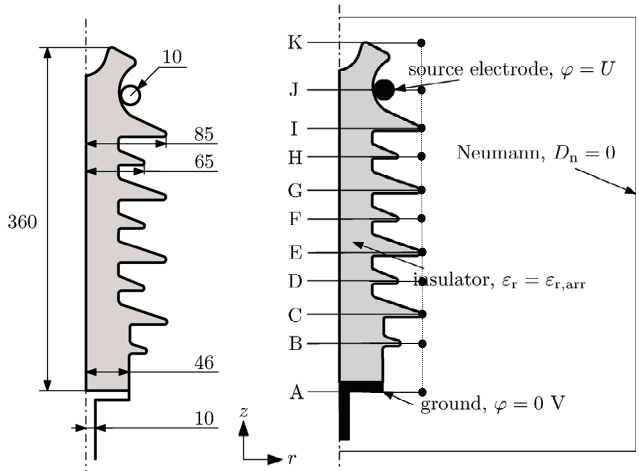

Fig. 1: Principal dimensions of the insulator (dimensions in mm) together with the boundary conditions

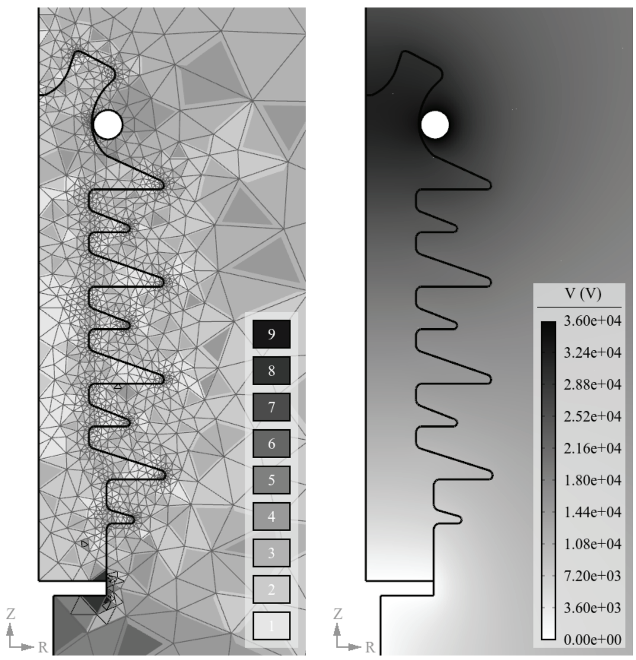

Fig. 2: Complete mesh after the adaptive process (left), distribution of electric potential φ (right)

The distributions of voltage and electric field strength along the dotted perpendicular line in the right part of Fig. 1 are shown in Figs. 4 and 5.

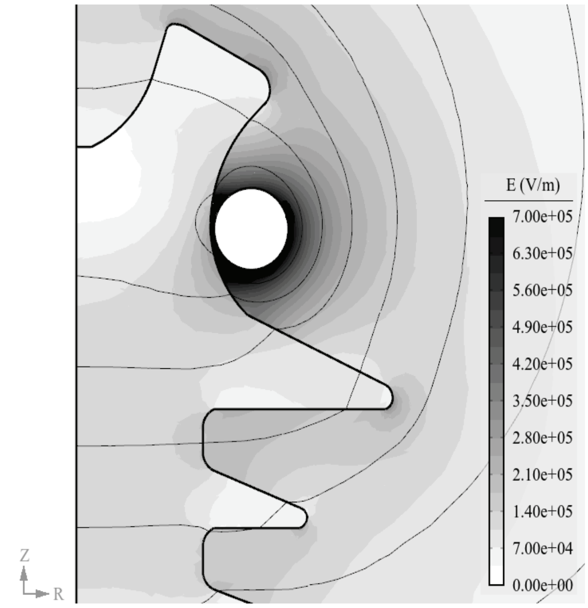

Fig. 3: Zoom of the upper part of the insulator and distribution of the corresponding part of electric field

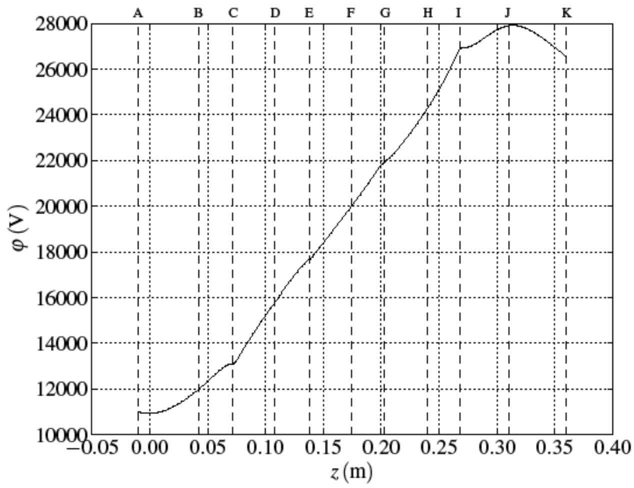

Fig. 4: Distribution of electric potential along the dotted perpendicular line in the right part of Fig. 1

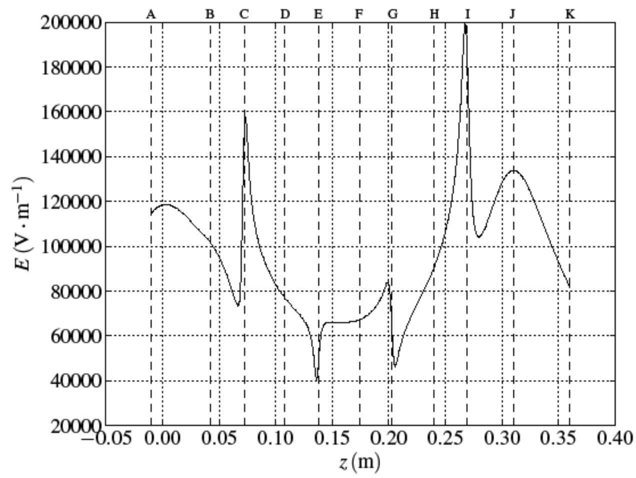

Fig. 5: Distribution of electric field strength along the dotted perpendicular line in the right part of Fig. 1

The most important convergence graphs are depicted in Fig. 6. Compared is the value of the total electrostatic energy W_e in the system as a function of the number of DOFs. The value of W_e is given by the integral

$$W_\mathrm{e} = \frac{\varepsilon_0}{2}\int_V \varepsilon \left|\boldsymbol{E}\right|^2 \mathrm{d}V \,, \tag{5}$$

where $\boldsymbol{E} = -\mathrm{grad}\varphi$, $\varepsilon_0 = 10^{-9}/\left(36\pi\right)$ F/m and V is the volume of the domain.

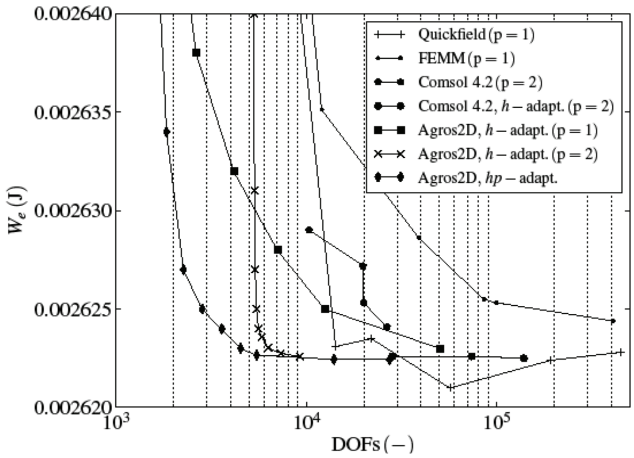

Fig. 6: Convergence curves of the total electrostatic energy W_e in the region

The codes FEMM and QuickField only work with linear elements without adaptivity and the results obviously converge very slowly. Faster is the convergence in Comsol Multiphysics. On the other hand, this code does not support the hanging nodes, so that much more elements are needed. Finally, Agros2D with adaptivity starting from a rough mesh converges extremely fast, with a substantially lower number of DOFs. Thus, usage of Agros2D with curved elements is much more effective.

Conclusion

The presented adaptive techniques prove to be extremely powerful in high-demanding computations based on the finite element methods of higher orders of accuracy. The number of DOFs necessary for the solution at a given accuracy can be even more than ten times lower compared with classical low-order methods of this type.

Acknowledgement

This work was supported by the European Regional Development Fund and Ministry of Education, Youth and Sports of the Czech Republic under the project No. CZ.1.05/2.1.00/03.0094 – RICE, by the Grant project P102/11/0498 (GACR) and project SGS-2012-039.

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