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Self inductance of long conductor of rectangular cross section

Abstract. *In this paper, the self inductance for long conductor with rectangular cross section is in investigated. Using three-dimensional integral Fredholm's equation of the second kind with weakly singular kernel we obtain equation for the complex voltage drop in the conductor. Self impedance appearing in the equation is expressed in the form of integral relation for any current density distribution. The imaginary part of this impedance divided by angular frequency is the self inductance of conductor of any shape and finite length. In case of direct current (DC), low frequency (LF) or thin strip long conductor of rectangular cross section the formulae for the self inductances are given for any length and for length much greater than the other dimensions. The self inductance of a thin tape is also presented.*

Streszczenie. *W pracy badano indukcyjność własną długiego przewodu o przekroju prostokątnym. Stosując trójwymiarowe równanie całkowe Fredholma drugiego rodzaju z jądrem słabo osobliwym otrzymano równanie na zespolony spadek napięcia w przewodzie. Występująca w tym równaniu impedancja własna jest wyrażona w postaci całkowej dla dowolnego rozkładu gęstości prądu. Część urojona tej impedancji dzielona przez częstotliwość kątową jest indukcyjnością własną przewodu o dowolnym przekroju poprzecznym i dowolnej długości. Wzory na indukcyjność własną długiego przewodu o przekroju prostokątnym i długości znacznie większej niż jego wymiary poprzeczne podano dla przypadku prądu stałego, przemiennego o niskiej częstotliwości lub przewodu taśmowego. (**Indukcyjność własna długiego przewodu o przekroju prostokątnym**)*

Key words: rectangular busbar, self inductance, electromagnetic field, integral equation
Słowa kluczowe: prostokątny przewód szynowy, indukcyjność własna, pole elektromagnetyczne, równanie całkowe

Introduction

Real lumped isolated conductor can be modeled as a connection, in series or in parallel, a resistance and an self inductance. The self inductance plays an important role not only in power circuits [1], but also in printed circuit board (PCB) lands [2, 3].

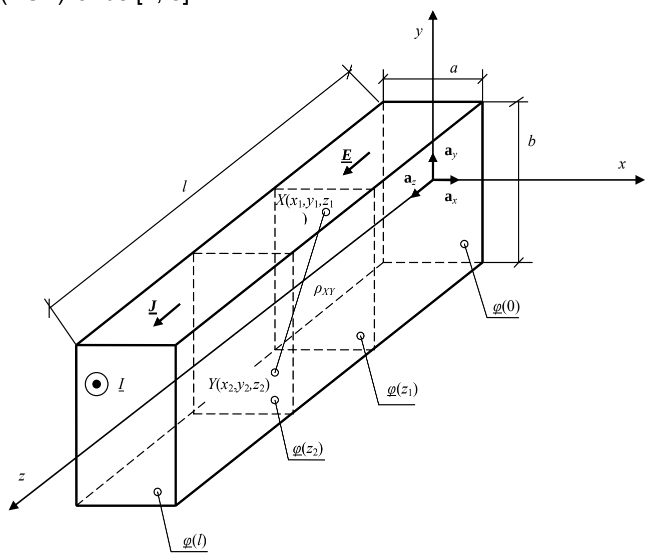

Fig. 1. A conductor of rectangular cross section with width *a*, thickness *b*, length *l* and current $\underline{I}$

Formulae for the self inductances of conductors of rectangular cross-section are the subjects of many electrical papers and books. They are mathematically complex and their demonstrations are usually omitted (not in [3] and [4]) and only the approximate formulae are given as though they were exact [4].

In general case there are two methods to calculate self inductance: the first one is calculation of inductance of a current-carrying closed loop and the second is calculation of induction of a segment of a current loop using the concept of partial inductance [3]. In this paper a new method for calculating self inductance is presented. The method results in integral Fredholm's equation. We compare our formulae with several well-known ones given in the literature.

There are many significant formulae for self inductance of conductor of rectangular cross section. In case of DC, power frequency or a thin rectangular conductor with width *a*, thickness *b*, and length *l* as shown in Fig. 1, Grover gives the following self inductance formula [2, 3, 5, 6]:

$$L = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{a+b} - 0.2235\ln\frac{a+b}{l} + \frac{1}{2}\right) \tag{1}$$

According to Kalantarov and Tseitlin [7] we have

$$L = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{a+b} + \frac{1}{2}\right) \tag{2}$$

Strunsky in [8] gives the formula

$$L = \frac{\mu_0 l}{2\pi}\left(\ln\frac{2l}{g} - 1 + \frac{g}{l}\right) \tag{3}$$

where *g* is the geometric mean distance from the rectangle to itself and is expressed by the formula

$$\begin{aligned}\ln g = {}& \frac{1}{2}\ln\left(a^2+b^2\right) - \frac{1}{12}\frac{a^2}{b^2}\ln\left(1+\frac{b^2}{a^2}\right) - \\ & \frac{1}{12}\frac{b^2}{a^2}\ln\left(1+\frac{a^2}{b^2}\right) + \frac{2}{3}\frac{a}{b}\tan^{-1}\frac{b}{a} + \frac{2}{3}\frac{b}{a}\tan^{-1}\frac{a}{b} - \frac{25}{12}\end{aligned} \tag{4}$$

The formula for self inductance of rectangular conductor given by Bueno and Assis in [9] is

$$\begin{aligned}L = {}& \frac{\mu_0 l}{4\pi}\left[2\ln(2l) - \frac{a^2}{3b^2}\ln a - \frac{b^2}{3a^2}\ln b - \left(1 - \frac{a^2}{6b^2} - \frac{b^2}{6b^2}\right)\right]\times \\ & \times\ln\left(a^2+b^2\right) - \frac{4}{3}\frac{a}{b}\tan^{-1}\frac{b}{a} - \frac{4}{3}\frac{b}{a}\tan^{-1}\frac{a}{b} + \frac{13}{6}\end{aligned} \tag{5}$$

Ruehli's formula is given in [3, 6, 10) while Hoer and Love's one in [3, 6, 11].

Definition of self inductance

The definition of self inductance of straight conductor is given in [12-14] by following formula

(6) $$L=\frac{1}{\omega}\operatorname{Im}\underline{Z}=\frac{\mu_0}{4\pi I^2}\int_{v_1}\int_{v_2}\frac{\underline{J}(Y)\underline{J}^*(X)}{\rho_{XY}}\,\mathrm{d}v_1\,\mathrm{d}v_2$$

where $\underline{J}(Y)$ is the complex current density at source point $Y\in$ S and Y=Y(x_2, y_2, z_2), $\underline{J}(X)$ is the complex current density at point of observation and $X\in \mathrm{R}^3$, v is conductor's volume and distance between the point of observation X and the source point Y (Fig.1) is given by the formula

(7) $$\rho_{XY}=\sqrt{r_{XY}^2+(z_2-z_1)^2}$$

where

(8) $$r_{XY}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

One can see from the formulas (6) that the inductance of the conductor depends on the distribution of current density in this conductor.

The formula (6) specifies the self-inductance of a solid conductor as a parameter standing next to jω in the equation

(9) $$\underline{U}=\underline{Z}\,\underline{I}=(R+\mathrm{j}\,\omega L)\underline{I}$$

which specifies the voltage drop and from the point of theory of circuits it is called [12-14] self-inductance. It may not be associated with a closed loop (according to the classical view of self-inductance of a closed circuit) but it should be merely considered as a quantity helpful in calculating the self-inductances of real closed electrical circuits.

If a conductor has a constant cross-sectional area S along its length and in case of DC, low frequency (for busbars of dimensions used in electrical power distribution system [1]) or for a very thin strip conductor (in printed circuit board [2, 3]) we can assume that the current density is constant and given as $\underline{J}(X)=\underline{I}/S$ and then, from the formula (6), we obtain the self-inductance of a straight conductor

(10) $$L=\frac{\mu_0}{4\pi S^2}\int_v\int_v\frac{1}{\rho_{XY}}\,\mathrm{d}v_1\,\mathrm{d}v_2$$

Self inductance of rectangular conductor

The self inductance of a rectangular conductor of dimensions $a\times b\times l$ shown in Fig. 1 is given by formula

(11) $$L=\frac{\mu_0}{4\pi}\frac{1}{a^2\,b^2}F$$

where

(12) $$F=\int_0^l\int_0^l\int_0^b\int_0^b\int_0^a\int_0^a\frac{1}{\rho_{XY}}\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z_1\mathrm{d}z_2$$

is a sixtuple definite integral of six variables. In general case this integral is very difficult to calculate. But if two variables, for example x_1 and x_2, can be replaced with only one variable $x=x_2-x_1$ then a double definite integral can be calculated from following formula

(13) $$F(y,z)=\int_{s_3}^{s_4}\int_{s_1}^{s_2}f(x_2-x_1,y,z)\,\mathrm{d}x_2\,\mathrm{d}x_1=$$
$$=F(s_4-s_1)-F(s_4-s_2)+F(s_3-s_2)-F(s_3-s_1)$$

or in more general form as

(14)
$$F(y,z)=\left[F(x,y,z)\right]_{s_4-s_2,\,s_3-s_1}^{s_4-s_1,\,s_3-s_2}(x)=\left[F(x,y,z)\right]_{p_2,\,p_4}^{p_1,\,p_3}(x)=\sum_{i=1}^{i=4}(-1)^{i+1}F(p_k,y,z)$$

where

(15) $$F(x,y,z)=\iint f(x,y,z)\,dxdx$$

is a double indefinite integral of $f(x,y,z)$. So in (12) we can also omit terms proportional to one variable like $F(x,y,z)=x\,g(y,z)$.

Hence the double definite integral

(16)
$$f(x,y)=\int_0^l\int_0^l\frac{1}{\rho_{XY}}\mathrm{d}z_1\mathrm{d}z_2=$$
$$2\,l\left(\ln\frac{l+\sqrt{l^2+r_{XY}^2}}{r_{XY}}-\frac{\sqrt{l^2+r_{XY}^2}}{l}+\frac{r_{XY}}{l}\right)$$

If $l>>r_{XY}$ the function $f(x,y)$ becomes

(17) $$f(x,y)=2\,l\left(\ln\frac{2\,l}{r_{XY}}-1\right)$$

and the self inductance of the rectangular conductor expresses by formula

(18) $$L=\frac{\mu_0\,l}{2\pi}\left[\ln(2\,l)-1+G\right]$$

where

(19)
$$G=-\frac{1}{2\,a^2b^2}\times$$
$$\times\int_0^b\int_0^b\int_0^a\int_0^a\ln\left[(x_2-x_1)^2+(y_2-y_1)^2\right]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2$$

Now we can put $x=x_2-x_1$ and $y=y_2-y_1$ and first calculate a quadruple indefinite integral

(20
$$G(x,y)=-\frac{1}{2\,a^2b^2}\iiiint\ln\left[x^2+y^2\right]\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}y=\frac{1}{288a^2b^2}\times$$
$$\left\{\begin{aligned}&150x^2y^2-\\&6\left[8\,x\,y^3\tan^{-1}\frac{x}{y}+8x^3y\tan^{-1}\frac{y}{x}+\left(x^4-6\,x^2y^2+y^4\right)\ln\left(x^2+y^2\right)\right]\end{aligned}\right\}$$

After calculating this integral we determine the self inductance of the long conductor of rectangular cross section

(21) $$L=\frac{\mu_0\, l}{2\,\pi}\left\{\ln(2\,l)-1+\left[\left[G(x,y)\right]\begin{matrix}a,-a\\(x)\\0,0\end{matrix}\right]\begin{matrix}b,-b\\(y)\\0,0\end{matrix}\right\}=\frac{\mu_0\, l}{2\,\pi}\left\{\ln(2\,l)-1+\sum_{i=1}^{i=4}\sum_{j=1}^{j=4}(-1)^{i+j}F(p_i,q_j)\right\}$$

On the basis of (21) we have the analytical formulae for the self inductance of the straight long conductor of rectangular cross section

(22) $$L=\frac{\mu_0 l}{2\pi}\left\{\ln\frac{2l}{a}+\frac{13}{12}-\frac{2}{3}\left[\frac{b}{a}\tan^{-1}\frac{a}{b}+\frac{a}{b}\tan^{-1}\frac{b}{a}\right]+\frac{1}{12}\left[\left(\frac{a}{b}\right)^2-6+\left(\frac{b}{a}\right)^2\right]\ln\left[1+\left(\frac{a}{b}\right)^2\right]+\frac{1}{6}\left[6-\left(\frac{a}{b}\right)^2\right]\ln\frac{a}{b}\right\}$$

or

(23) $$L=\frac{\mu_0 l}{2\pi}\left\{\ln\frac{2l}{b}+\frac{13}{12}-\frac{2}{3}\left[\frac{b}{a}\tan^{-1}\frac{a}{b}+\frac{a}{b}\tan^{-1}\frac{b}{a}\right]+\frac{1}{12}\left[\left(\frac{a}{b}\right)^2-6+\left(\frac{b}{a}\right)^2\right]\ln\left[1+\left(\frac{b}{a}\right)^2\right]+\frac{1}{6}\left[6-\left(\frac{b}{a}\right)^2\right]\ln\frac{b}{a}\right\}$$

as well as

(24)
$$L=\frac{\mu_0 l}{2\pi}\left\{\ln\frac{2l}{a+b}+\frac{13}{12}-\frac{2}{3}\left[\frac{b}{a}\tan^{-1}\frac{a}{b}+\frac{a}{b}\tan^{-1}\frac{b}{a}\right]+\frac{1}{2}\ln\left[1+\frac{a}{b}\frac{2}{1+\left(\frac{a}{b}\right)^2}\right]+\frac{1}{12}\left[\left(\frac{a}{b}\right)^2\ln\left[1+\left(\frac{b}{a}\right)^2\right]+\left(\frac{b}{a}\right)^2\ln\left[1+\left(\frac{a}{b}\right)^2\right]\right]\right\}$$

For the chosen traverse dimensions and different lengths of a busbar the calculations of its inductance have been made according to all previous, shown above, formulae – Table 1.

Table 1. Self-inductance of a busbar of rectangular cross section for DC or low frequency

Busbar: $a=0.08\,\mathrm{m}$; $b=0.007\,\mathrm{m}$								
	Eq. (1)	Eq. (2)	Eq. (3)	Eq. (5)	Ruehli	Hoer	Eq. (11)	Eq. (22)
l (m)	*L* (nH)	*L* (nH)	*L* (nH)	*L* (nH)	*L* (nH)	*L* (nH)	*L* (nH)	*L* (nH)
0.01 *a*	negative	negative	3.33051	negative	0.00567	0.00567	0.00567	negative
0.10 *a*	negative	negative	1.97768	negative	0.46616	0.46616	0.46616	negative
1.00 *a*	13.8593	17.7482	21.6067	17.7165	22.4732	22.4732	22.4732	17.7165
10.0 *a*	542.007	545.896	549.469	545.579	550.911	550.911	550.911	545.579
100 *a*	9139.20	9143.09	9143.81	9139.92	9145.32	9145.32	9145.32	9139.92
1000 *a*	128268	128272	128244	128240	128227	128294	128217	128240

Self inductance of thin tape

The self inductance of a thin tape of width *a*, thickness *b*≈0 and length *l* is given by formula

(25) $$L=\frac{\mu_0}{4\pi}\frac{1}{a^2}F$$

where

(26) $$F=\int_0^l\int_0^l\int_0^a\int_0^a\frac{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}z_1\mathrm{d}z_2}{\sqrt{(x_2-x_1)^2+(z_2-z_1)^2}}$$

is a quadruple definite integral of four variables (x_1,x_2,z_1,z_2). Now we can put $x=x_2-x_1$ and $z=z_2-z_1$ and first calculate a quadruple indefinite integral

(27) $$F(x,z)=\iint\iint\frac{\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}z}{\sqrt{x^2+z^2}}$$

twice with respect to *x* and twice with respect to *z*.

Finally, after a lengthy integration, formula (27) yields an expression for quadruple indefinite integral

(28) $$F(x,z)=\frac{1}{2}\left[\begin{matrix}-\frac{3}{2}z\left(x^2+z^2\right)-\frac{1}{3}\left(x^2+z^2\right)^{3/2}+\\ xz^2\ln\left(x+\sqrt{x^2+z^2}\right)+x^2z\ln\left(z+\sqrt{x^2+z^2}\right)\end{matrix}\right]$$

Hence the self inductance of the thin tape is given by following formula

(29) $$L=\frac{\mu_0}{4\pi}\frac{1}{a^2}\left[\left[F(x,z)\right]\begin{matrix}a,-a\\(x)\\0,0\end{matrix}\right]\begin{matrix}l,-l\\(z)\\0,0\end{matrix}=\frac{\mu_0}{4\pi}\frac{1}{a^2}\sum_{i=1}^{i=4}\sum_{k=1}^{k=4}(-1)^{i+k}F(p_i,r_k)$$

On the basis of (29) we have an analytical formula for the self inductance of the thin tape

$$L = \frac{\mu_0}{6\pi a^2}\left[3a^2 l \ln\frac{l+\sqrt{l^2+a^2}}{a} - \left(l^2+a^2\right)^{3/2} + 3al^2\ln\frac{a+\sqrt{l^2+a^2}}{l} + l^3 + a^3\right] \tag{30}$$

It is exactly the Hoer and Love's formula given in [11].

For the self inductance of a thin tape of width *a*, thickness *b* and length *l* above formulae give results shown in Table 2.

Table 2. Self-inductance of a thin tape of rectangular cross section for DC or low frequency

Thin tape: $a = 0.5\ \mu$m; $b = 0.1\ \mu$m								
	Eq. (1)	Eq. (2)	Eq. (3)	Eq. (5)	Ruehli	Hoer	Eq. (11)	Eq. (30)
l (m)	*L* (pH)	*L* (pH)	*L* (pH)	*L* (pH)	*L* (pH)	*L* (pH)	*L* (pH)	*L* (pH)
0.01 *a*	negative	negative	0.02324	negative	0.000281	0.00002	0.00002	0.00005
0.10 *a*	negative	negative	0.01389	negative	0.002473	0.00247	0.00247	0.00352
1.00 *a*	0.00742	0.01010	0.01476	0.10083	0.131661	0.13166	0.13166	0.14866
10.0 *a*	3.28659	3.31341	3.33776	3.31092	3.345462	3.34546	3.34546	3.52865
100 *a*	56.1331	56.1599	56.1618	56.1351	56.16998	56.1699	56.1699	58.0165
1000 *a*	791.831	791.858	791.635	791.609	791.6432	791.634	791.337	810.124

Conclusions

In this paper we have presented a new method for calculating of self inductance of an isolated conductor of rectangular cross section. Using three-dimensional integral Fredholm's we have defined the self inductance of conductor of any shape and finite length given by sixtuple definite integral. In case of DC or low frequency we have given a general formulae for self inductance of conductors of rectangular cross section of any dimensions including the thin tapes and "very long" ones. By computations we have shown that our formulae give the same results as first of all Hoer's ones for all dimensions of conductor. But experimental results show that our formulae are numerically more stable and accurate for all dimensions of conductor than the others.

In addition we have also obtained analytical forms of all formulae which are more useful than general ones. Of course they give the same results than the general formulae.

Our formulae are analytically simple and can also replace the traditional tables and working ones.

These formulae can be used in the methods of numerical calculation of AC self inductance of rectangular conductor. Then the cross section of the conductor is divided into rectangular subbars (elementary bars) in which the current is assumed to be uniformly distributed over the cross section of each subbars.

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REFERENCES

[1] Balzer G. et al: *Switcher Manual.* 10th edn., ABB Calor Emag Mittelspannung GmbH, Ratingen 2004.

[2] Kazimierczuk M. K.: *High-Frequency Magnetic Components.* J Wiley & Sons, Chichester, 2009.

[3] Paul C. R.: *Inductance: Loop and Partial.* J Wiley & Sons, New Jersey, 2010.

[4] Rosa E. B.: *The Self and Mutual Inductances of Linear Conductors*. Bul. NBS., No. 4, 1908, pp. 301- 344.

[5] Grover F. W. *Inductance Calculations.* Dover Publications, Inc., New York, 1973.

[6] Kim H. and Chen C. C. P.: *Be Careful of Self and Mutual Inductance Formulae.* Online: http:// ccf.ee.ntu.edu.tw/~cchen/research/CompInduct9.pdf), 2009.

[7] Kalantarov P. L. and Tseitlin L. A.: *Inductance Calculations* (in Russian). Energiya, Saint Petersburg 1970.

[8] Strunsky B. M.: *Short Electric Network of Electric Furnaces* (in Russian). GN-TIL, Moskow 1962.

[9] Bueno M. A. and Assis A. K. T. *A new Method for Inductance Calculations*. J. Phys. D: Appl. No. 28, 1995, pp. 1802-1806.

[10] Ruehli A. E.: *Inductance Calculations in a Complex Integrated Circuit Environment*. IBM J. Res. Develop. Sept. 1972, pp. 470-481.

[11] Hoer C. and Love C.: *Exact Inductance Equations for Rectangular Conductors with Application to More Complicated Geometries*. J. Res. N. B. S., No. 2, 1965, pp. 127-137.

[12] Piątek Z.: *Inductance of Any Length Busbar of Recatangular Cross Section* (in Polish). Przegląd Elektrotechniczny, No. 2, 2007, pp. 119-122.

[13] Piatek Z.: *Self and Mutual Impedances of a Finite Length Gas Insulated Transmission Line (GIL)*. Elec. Pow. Syst. Res., No. 77, 2007, pp. 191-203.

[14] Piątek Z.: *Impedances of Tubular High Current Busducts*. Polish Academy of Sciences. Warsaw 2008.

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