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Effects of Local Volume Constraints on Optimal Topologies of Continuums

***Abstract**. Optimal stiffness design of a structure with local volume constraints on its subdomains is investigated. In practical engineering, a structure may have many subdomains with local volume constraints to meet the multi-function of structure. A new heuristic approach simulating the bone remodelling process is presented to solve such problem. The essentials of the present approach are summarized as follows. Firstly, the topology optimization of structure is equivalent to bone remodelling process. Corresponding to the dead zone in bone remodelling theory, a floating interval of reference strain energy density (SED) is proposed. Secondly, the update of the design variable, i.e. the relative density of a material point, is determined by comparison between the local SED and the current interval of reference SED. Thirdly, to satisfy the global constraints in an optimization problem, the global reference interval changes in simulation. Finally, to satisfy the local volume constraints of subdomains in structure, the same amount of local reference intervals are adopted to modify the update rule of local materials. Numerical examples are employed to demonstrate the effects of the local volume constraints on the optimal topologies of structures.*

***Streszczenie.** Zbadano metody optymalnego projektowania system z ograniczeniami lokalnych rozmiarów w subdomenach. Przedstawiono nową metodę heurystyczną symulującą szkielet procesu modelowania w celu rozwiązania tego problemu (**Efekt ograniczeń lokalnych wielkości w optymalnej topologii kontinuum**)*

Keywords: Topology optimization, Local volume constraint, Reference interval, SED.
Słowa kluczowe: lokalne ograniczenie wielkości, optymalizacja topologii

1. Introduction

To give an optimal stiffness design of a continuum structure, topology optimization is an important mean in the conceptual design phase of the structure [1]. The reason is that traditional shape or size optimization can not change the topology of a structure during the solution process. During the past two decades, theories of topology optimization methods have been developed, rapidly. Briefly, Eschenauer and Olhoff [2] classified those methods into two types, i.e. the material methods and the geometry methods [3-6].

For a traditional stiffness design of a continuum structure global constraints, e.g., volume constraint of structure, the displacement constraint and/or the stress constraint on a region of structure, has been investigated for a long time and can be solved successfully. However, a structure may have many subdomains and some of the subdomains have volume constraints to meet the multi-function of structure, such as being channels for fluid or cables or as frames to support local concentrated loadings. The local volume constraint influence should be considered together with other constraints in the optimization of a structure with subdomains. Obviously, to solve such kind of optimization problem is significant in engineering. But little effort has been taken to solve such kind of problem previously.

In the present work, a simply bionics approach is presented to solve this kind of topology optimization problem. In optimization process, the update rule of design variables is performed by an intuitive evolutionary method based on bone remodeling theories [7-10], rather than by a mathematical programming approach.

2. Material Properties

2.1. Definitions of elastic tensor

In porous materials, the mechanical properties are related closely to their micro-structures. Efforts have been taken to establish theoretical foundation to characterize the accurate relationships between the anisotropy and the microstructure [11-14]. It was found that micro-structural properties can be described as an invariant form by a set of even rank fabric tensors [14] and can be assessed accurately using stereological methods [15]. They concluded that the principal directions of the fabric tensor coincide with those of orthotropic elastic tensor. In most applications, orthotropy material properties seem to be sufficiently well described by a symmetric, positive and definite second rank fabric tensor [13]. To express the material properties, here we adopt the approach suggested by Zysset and Curnier [13].

2.2. Stiffness tensor

From an experimental point of view, the component matrix of anisotropic elasticity of a material can be identified by using two independent material constants (λ_0, μ_0), a second rank fabric tensor and an exponent (ω) [13]

$$D_{0,ijkl} = \lambda_0 B_{ij}^{\omega} B_{kl}^{\omega} + \mu_0 \left(B_{ik}^{\omega} B_{jl}^{\omega} + B_{il}^{\omega} B_{jk}^{\omega} \right) \tag{1}$$

B_{ij} is the component matrix of fabric tensor. In this work, a particular elasticity model with $\lambda_0 = \lambda, \mu_0 = \mu$ and $\omega = 1.5$ in Eq. (1) is adopted. Clearly, $D_{0,ijkl}$ in Eq.(1) expresses an isotropic material when B_{ij} is proportional to the Kronecker delt (δ_{ij}), i.e., $B_{ij} = \rho \cdot \delta_{ij}$. Meanwhile, $D_{0,ijkl}$ is expressed as

$$D_{0,ijkl} = \rho^3 \left[\lambda \, \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \right] \tag{2}$$

where ρ is the volume fraction of solid in porous material.

3. Optimization model

3.1. Floating interval of reference SED method

For the current floating reference interval method, the formulations of topology optimization can be expressed as

$$\begin{aligned} &\text{Find} && \{ \rho_m \} \\ &\text{to satisfy} && u_m \in \left[u_{\inf}^{ref} \quad u_{\sup}^{ref} \right] \\ &\text{subject to} && \phi_j \left(\{ \rho_m \} \right) \le 0, \left(j = 1, 2, \cdots \right) \\ &&& K \cdot U = P \\ &&& \rho_m \in \left[\rho_{\min} \quad 1.0 \right] \end{aligned} \tag{3}$$

where ρ_m is the design variable of the *m*-th material point (or element). u_m is the SED of the *m*-th material point, i.e., half of the scalar product of stress tensor and strain tensor. $\left[u_{\inf}^{ref} \quad u_{\sup}^{ref} \right]$ is the final interval of reference SED, or the global reference interval. ϕ_j is the *j*-th constraint function. $\rho_{\min}$ is set to be 0.001 to keep fabric tensor to be positive and definite. *K* is the global stiffness matrix of structure in finite element (FE) analysis. *U* and *P* are the global nodal displacement and nodal force vectors, respectively.

3.2. Present topology optimization formulations

(a) As the whole structure has a displacement constraint, the formulations for such optimization problem is constructed as

(4)
$$\begin{aligned}&\text{Find} && \{\rho_m\}\\ &\text{to satisfy} && u_m \in \left[u_{\inf}^{ref} \quad u_{\sup}^{ref}\right]\\ &&& u_m \in \left[{}_{sub}^{ki}u_{\inf}^{ref} \quad {}_{sub}^{ki}u_{\sup}^{ref}\right], \left(ki = 1, 2, \cdots, ks\right)\\ &\text{subject to} && d - d_0 = 0\\ &&& \sum_m v_m \xi_m^{ki} = V_{sub}^{ki}, \left(ki = 1, 2, \cdots, ks\right)\\ &&& K \cdot U = P\\ &&& \rho_m \in \left[\rho_{\min} \quad 1.0\right], \left(i = 1, 2, 3\right)\end{aligned}$$

where $\left[{}_{sub}^{ki}u_{\inf}^{ref} \quad {}_{sub}^{ki}u_{\sup}^{ref}\right]$ is the interval of reference SED for the *ki*-th subdomain or the the ki-th local reference interval. d_0 is the critical value of displacement constraint. v_m is the amount of solid phase of the m-th material point. V_{sub}^{ki} is the critical value of the volume constraint on the *ki*-th subdomain. When the m-th material point locates in the ki-th subdomain, ξ_{sub}^{ki} equals 1. Otherwise, ξ_{sub}^{ki} equals 0.

(b) As the whole structure has a volume constraint, the formulations for such kind of optimization problem can be expressed as follows

(5)
$$\begin{aligned}&\text{Find} && \{\rho_m\}\\ &\text{to satisfy} && u_m \in \left[u_{\inf}^{ref} \quad u_{\sup}^{ref}\right]\\ &&& u_m \in \left[{}_{sub}^{ki}u_{\inf}^{ref} \quad {}_{sub}^{ki}u_{\sup}^{ref}\right], \left(ki = 1, 2, \cdots, ks\right)\\ &\text{subject to} && \sum_m v_m - V_0 = 0\\ &&& \sum_m v_m \xi_m^{ki} - V_{sub}^{ki} = 0, \left(ki = 1, 2, \cdots, ks\right)\\ &&& K \cdot U = P\\ &&& \rho_m \in \left[\rho_{\min} \quad 1.0\right]\end{aligned}$$

where V_0 is the critical value of the volume constraint on the whole structure.

3.3. Update rule of design variables

According to the concept of dead zone in bone remodeling theory [8], we introduce the following update rule. In order to obtain the optimal topology of a structure, a state of remodeling equilibrium which requires the local SED of each material point within the admissible design domain being in an interval of reference SED [10], should be reached. According to the above rule, the material distribution within the design domain will be changed in simulation if the local SED is out of the current reference interval. Mathematically, the increment of the relative density of a material point, which locates in the *ki*-th subdomain, can be expressed as

(6)
$$\Delta\rho_{k,m} = \begin{cases} g_1 > 0, & u_{k,m} > {}_{\text{sub}}^{ki}u_{\sup}^{ref(k-1)} \\ 0 & \text{others} \\ -g_2 < 0, & u_{k,m} < {}_{\text{sub}}^{ki}u_{\inf}^{ref(k-1)} \end{cases}$$

where $\Delta\rho_{k,m}$ is the increment of the relative density of the *m*-th material point at the *k*-th iteration step. g_1, the deposition speed, is set to be 0.09. g_2, the dissipation speed, is 0.06, in the present work.

For the update of the relative densities of the material points in the rest of the structure can be illustrated as follows

(7)
$$\Delta\rho_{k,m} = \begin{cases} g_1 > 0, & u_{k,m} > u_{\sup}^{ref(k-1)} \\ 0 & \text{others} \\ -g_2 < 0, & u_{k,m} < u_{\inf}^{ref(k-1)} \end{cases}$$

Therefore, the relative density of a material point in design domain can be updated as follows

(8)
$$\rho_{k+1,m} = \begin{cases} 1.0, & (\rho_{i,k,m} + \Delta\rho_{k,m}) \geq 1.0 \\ \rho_{k,m} + \Delta\rho_{k,m}, & \text{others} \\ \delta, & (\rho_{i,k,m} + \Delta\rho_{k,m}) \leq \delta \end{cases}$$

Correspondingly, the stiffness tensor of the material point can be expressed as follows

(9)
$$D_{m,ijkl} = \rho_{k+1,m}^3 \left[\lambda\,\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right]$$

3.4. Update rules of the intervals of reference SED

To satisfy the constraints in optimization problem, the SED reference interval needs upated. Currently, $u_{\inf}^{ref} = u_{\sup}^{ref} = u^{ref}$ is suggested to keep SED uniformly in structure. Therefore, only the supremum of interval needs updated. Meanwhile, the update rule of the reference interval depends on the type of active constraint.

(10)
$$R = \left|\frac{H^k}{H_0}\right|^{\alpha}$$

where H_0 is the critical value (specified before simulation) of the active constraint (e.g., volume constraint, displacement constraint). H^k is the current value in the *k*-th iteration step. For example, if an optimization problem with volume constrain, H_0 is the specified volume for the final structure and H^k is the current volume after the *k*-th optimization of structure. The exponent α is positive when the volume constraint acts as the active constraint and is negative for displacement constraint. In the present work, $|\alpha|$ equals 1.0.

3.5. Update of the global reference interval

In an optimization, the global reference interval is updated firstly according to the active constaint on the whole structure. The update of the supremum of the reference interval is as follows

(11)
$$u_{k+1}^{ref} = \begin{cases} R^{\beta} \cdot u_k^{ref} & \text{if } R \geq 1.0 - \eta \\ R^{\gamma} \cdot u_k^{ref} & \text{if } R < 1.0 - \eta \end{cases}$$
$$\text{Mod}\left(k, i_{FEA}\right) = 0$$

Where the exponents β and γ are limited in simulation, e.g. $\beta \in \left[1.0 \quad 2.0\right]$, $\gamma \in \left[10 \quad 30\right]$. η, the algorithm tolerance, is set to be 0.001. Integer $i_{FEA} \in \{3, 4, 5\}$ is adopted.

3.6. Update of the local reference intervals

As the subdomains only have volume constraints, the update of the local reference intervals is determined by their volume constraints. ${}_{sub}^{ki}u_{\inf}^{ref} = {}_{sub}^{ki}u_{suf}^{ref} = \theta_{ki} \cdot u^{ref}$, (*ki* =1, 2, …, *ks*) is used. Ratio θ_{ki} changes with respect to the volume of the *ki*-th subdomain.

$$
\theta_{ki}^{(k+1)} = \begin{cases} \left[V_{ki}^{(k)} / V_{ki}^{0}\right]^{\beta} \cdot \theta_{ki}^{(k)} & if\ R \geq 1.0-\eta \\ \left[V_{ki}^{(k)} / V_{ki}^{0}\right]^{\gamma} \cdot \theta_{ki}^{(k)} & if\ R < 1.0-\eta \end{cases} \tag{12}
$$

$$
\mathrm{Mod}\left(k, i_{FEA}\right) = 0
$$

Where $V_{ki}^{(k)}$ is the current volume of the *ki*-th subdomain at the *k*-th iteration step. V_{ki}^{0} is the critical value of the *ki*-th subdomain. The initial value of the ratio, i.e. θ_{ki}^{0}=1.0.

3.7. Optimization procedure for the present approach

Step 1: Discretize the structure with finite elements and initiate parameters and let *k*=1;

Step 2: Obtain the strain and stress fields by using FE analysis, calculate the local SED of each element;

Step 3: Update the relative density (Eq.(8)) of each element, renew the global and local reference intervals (Eqs.(11) and (12));

Step 4: Determine iteration criterion: if the convergent conditions (in Eq. (13)) are satisfied **or** *k* is equal to a given maximum number of iteration, then go to ***Step 5***, otherwise let *k*=*k*+1 and go to ***Step 2***;

Step 5: Stop.

In the initial design, all the relative densities are set to be unity over the admissible design domain. The initial supremum of the global reference interval is set to be equal to the average SED of the initial structure under the given loading conditions. The convergent criteria in ***Step 4*** is expressed as

$$
\begin{cases} \left|R-1.0\right| \leq \eta & R \in \left\{R_H^{(j)},\ j = i_{FEA}-M, \cdots, i_{FEA}-1, i_{FEA}\right\} \\ \left|\theta_{ki}^{(j)} - 1.0\right| \leq \eta_{ki} = \eta & \left(ki = 1, 2, ..., ks\right) \end{cases} \tag{13}
$$

where integer *M* is no less than 2.

4. Numerical examples

Financial software ANSYS (v12.0) is adopted for FE analysis.

4.1. Example 1-A beam with many subdomains

Fig. 1 shows the initial design domain of a simply-supported sheet beam, which is made of three unit square subdomains. A concentrated force ***P*** is applied on the centre of the upper surface, vertically. Only one isotropic material with Poisson's ratio of 0.3 exists in structure. The objective is to minimize the structural compliance with global volume constraint ratio of 25%. Two cases are considered to find the difference between the final material distributions with or without local volume constraint.

(a) only global volume constraint on structure is considered;

(b) Besides global volume constraint, subdomain 1 and subdomain 3 are specified to have the same local volume ratio, with local critical volume ratio of 21%.

Fig. 2a gives the optimal material distribution of the beam with only global volume constraint. The materials distributes very near the site subjected to concentrated forece. Fig. 2b shows the optimal topology of the beam with both of global and local volume constraints. The material in subdomain 2 distributes loosely. Clearly, the topologies are different in subdomain 2 for two cases.

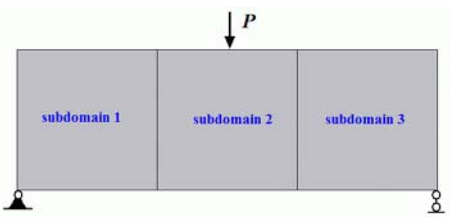
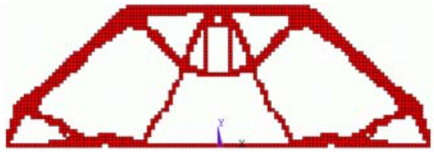


Fig. 1. Initial design domain of structure

(a) without local volume constraint



(b) with local volume constraints

Fig. 2. Optimal material distributions in structure with or without considering local volume constraints

4.2. Example 2-Cantilever beam

The design domain shown in Fig. 3 is a deep cantilever beam with size of 2.0m by 1.26m and the thickness is of 0.001m. The concentrated force, *P*=1.0kN, vertically acts on the centre (point C) of the right side. A rectangular subdomain with size of 0.48m by 0.5m exists in the middle of the structure. Outside of the subdomain, the frame with thickness of 0.02m is fixed, i.e., the material of frame keeps unchanged in simulation. The elastic modulus and the Poisson's ratio of the material outer of the subdomain are 21 GPa and 0.2, respectively.

The objective is to minimize the structural compliance. The critical value of the local volume ratio of the subdomain is specified to be 50%. Two cases are investigated:

(a) The whole structure has only one material and has displacement constraint, i.e., the displacement of point C reaches 0.001m;

(b) The whole structure has displacement constraint, i.e., the displacement of point C reaches 0.001m. In the subdomain, the elastic modulus and Poisson's ratio of material are 69 GPa and 0.3, respectively.

Fig. 4 gives the optimal material distributions in beam with displacement constraint on point C for cases (a) and (b). Obviously, the material distributes differently between the results of case (a) and (b).

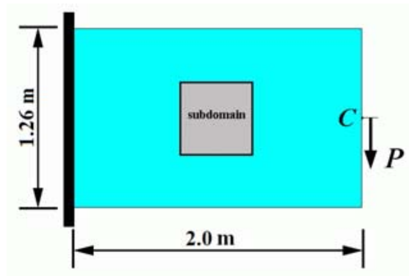
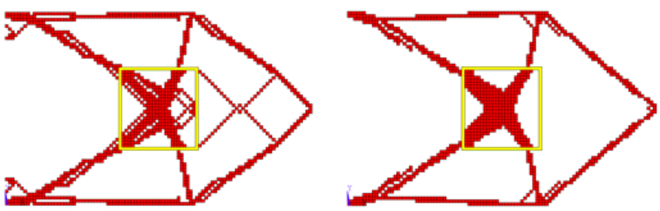
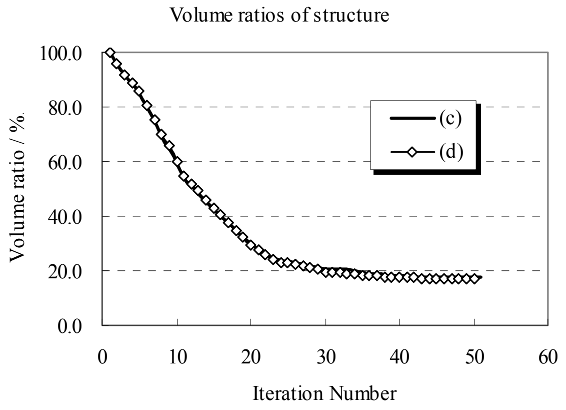


Fig. 3. Initial design domain of beam

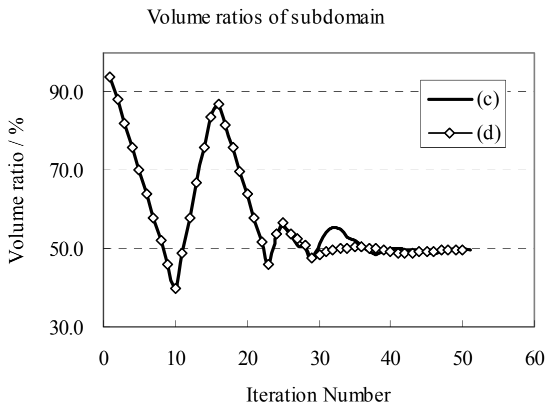


(a) results for case (c) (b) results for case (d)

Fig. 4. Optimal material distributions in beam with displacement constraint on point ***C***



(a) structural volume ratio



(b) iterations of local volume ratios

Fig. 5. Iteration histories of the volume ratios of both the whole structure and the subdomain with displacement constraint on point ***C***

Fig. 5 implies the iteration histories of the volume ratios of both the whole structure and the subdomain when the structure has displacement constraint on point ***C.*** For case (a), the structural volume ratio reaches 17.5% after 51 iterations. The structural volume ratio reaches 17.1% for case (b) after 50 times of iteration.

5. Conclusions

In practical engineering, the stiffness design of a structure may have many volume constrained subdomains for the purpose of multi-function. A bionics method is developed in the present work to solve such problem. The validity of the method is verified by numerical examples. The results of numerical examples also show that the optimal material distributions of a structure with or without considering local volume constraints on its subdomains are different. Considering further manufacturing, the local volume constraints must be considered in topology optimization.

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