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Zeroing of state variables in descriptor electrical circuits by state-feedbacks

***Abstract**. The problem of zeroing of the state variables in descriptor electrical circuits by state-feedbacks is formulated and solved. Necessary and sufficient conditions for the existence of gain matrices such that the state variables of closed-loop systems are zero for time greater zero are established. The procedure of choice of the gain matrices is demonstrated on simple descriptor electrical circuits with regular pencils.*

***Streszczenie.** Sformułowano i rozwiązano nowy problem zerowania zmiennych stanu deskryptorowych (singularnych) liniowych obwodów elektrycznych poprzez dobór odpowiednich sprzężeń zwrotnych. Podano warunki konieczne i wystarczające istnienia takich sprzężeń zwrotnych zapewniających zerowanie zmiennych stanu dla chwil czasowych większych od zera. Zaproponowano procedurę doboru sprzężeń zwrotnych, którą zilustrowano prostymi przykładami odwodów elektrycznych. (**Zerowanie zmiennych stanu deskryptorowych obwodów elektrycznych poprzez sprzężenia zwrotne**).*

Keywords: descriptor, linear, electrical circuit, state-feedbacks, zeroing of state variables.
Słowa kluczowe: deskryptorowy, liniowy, obwód elektryczny, sprzężenie zwrotne, zerowanie zmiennych stanu.

Introduction

Descriptor linear systems with regular pencils have been considered in many papers and books [1-5, 7-9, 18-20]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [6-7, 18] and the realization problem for singular positive continuous-time systems with delays in [11]. The computation of Kronecker's canonical form of a singular pencil has been analyzed in [20]. Luenberger in [19] has proposed the shuffle algorithm to analysis of the singular linear systems. A method for the checking of positivity of descriptor linear systems by the use of the shuffle algorithm has been proposed in [13]. The positivity and reachability of fractional electrical circuits have been addressed in [10, 14] and descriptor(singular) fractional linear systems and electrical circuits in [17]. Modified version of the shuffle algorithm has been proposed for the reduction of the singular fractional system into dynamic and static parts in [15]. The descriptor fractional discrete-time and continuous-time linear systems have been investigated in [16].

In this paper the problem of zeroing of the state variables in descriptor electrical circuits by state-feedbacks will be formulated and solved. The paper is organized as follows. In section 2 the descriptor linear electrical circuits are presented. The zeroing problem is formulated and solved in section 3 where the necessary and sufficient conditions for the existence of solution to the problem are established. Concluding remarks are given in section 4.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, I_n - the $n\times n$ identity matrix.

Descriptor linear electrical circuits

Consider the descriptor (singular) linear continuous-time system

(1) $E\dot{x} = Ax + Bu$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$ are the state and input vectors, respectively and $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.
It is assumed that $\det E = 0$, $\mathrm{rank} B = m$ and the pencil is regular, i.e.

(2) $\det[Es - A] \neq 0$ for some $s \in C$

(the field of complex numbers).

Example 1. Consider electrical circuit shown on Fig. 1 with given resistance R, capacitances C_1, C_2, C_3 and source voltages e_1 and e_2.

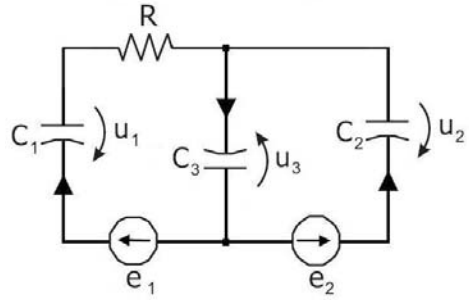

Fig. 1. Electrical circuit of Example 1.

Using Kirchhoff's laws, for the electrical circuit we can write the equations

$$e_1 = RC_1\frac{du_1}{dt} + u_1 + u_3,$$

$$C_1\frac{du_1}{dt} + C_2\frac{du_2}{dt} - C_3\frac{du_3}{dt} = 0,$$

(3) $e_2 = u_2 + u_3.$

The equations (3) can be written in the form

(4) $$\begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix}\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

In this case we have

(5) $$E = \begin{bmatrix} RC_1 & 0 & 0 \\ C_1 & C_2 & -C_3 \\ 0 & 0 & 0 \end{bmatrix},\quad A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix},\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that the matrix E is singular ($\det E = 0$) but the pencil

(6) $$\det[Es - A] = \begin{vmatrix} RC_1 s + 1 & 0 & 1 \\ C_1 s & C_2 s & -C_3 s \\ 0 & 1 & 1 \end{vmatrix} = (RC_1 s + 1)(C_2 + C_3)s + C_1 s$$

is regular. Therefore, the electrical circuit is a descriptor linear system with regular pencil.
In general case we have the following theorem.
Theorem 1. If the electrical circuit contains at least one mesh consisting of branches with only ideal capacitors and voltage sources, then its matrix E is singular.
Proof. Note that the row of E corresponding to the mesh is a zero row. This follows from the fact that the equation written with the use of Kirchhoff's voltage law is an algebraic one.□
Example 2. Consider electrical circuit shown on Fig. 2 with given resistances R_1, R_2, R_3 inductances L_1, L_2, L_3 and source voltages e_1 and e_2.

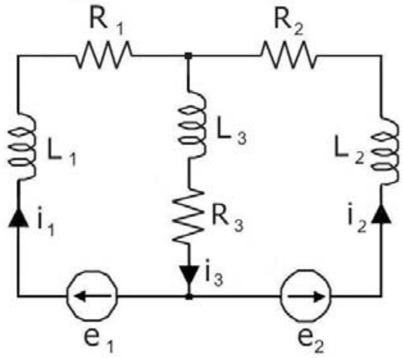

Fig. 2. Electrical circuit of Example 2.2.

Using Kirchhoff's laws we can write the equations

$$e_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + R_3 i_3 + L_3 \frac{di_3}{dt},$$

$$e_2 = R_2 i_2 + L_2 \frac{di_2}{dt} + R_3 i_3 + L_3 \frac{di_3}{dt},$$

(7) $$i_1 + i_2 - i_3 = 0.$$

Equations (7) can be written in the form

(8) $$\begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

In this case we have

(9) $$E = \begin{bmatrix} L_1 & 0 & L_3 \\ 0 & L_2 & L_3 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & 0 & -R_3 \\ 0 & -R_2 & -R_3 \\ 1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Note that the matrix E is singular but the pencil

(10)
$$\det[Es - A] = \begin{vmatrix} L_1 s + R_1 & 0 & L_3 s + R_3 \\ 0 & L_2 s + R_2 & L_3 s + R_3 \\ -1 & -1 & 1 \end{vmatrix} = [L_1(L_2 + L_3) + L_2 L_3]s^2$$
$$+ [(L_1 + L_3)R_2 + (L_2 + L_3)R_2 + (L_2 + L_3)R_3]s + R_1(R_2 + R_3) + R_2 R_3$$

is regular. Therefore, the electrical circuit is a descriptor linear system with regular pencil.
Theorem 2. If the electrical circuit contains at least one node with branches with coils then its matrix E is singular.
Proof. Note that the equation written using the current Kirchhoff's current law for this node is an algebraic one and in the matrix E we have zero row. □
In general case we have the following theorem.
Theorem 3. Every electrical circuit is a singular system if it contains at least one mesh consisting of branches with only ideal capacitances and voltage sources or at least one node with branches with coils.
Proof. By Theorem 1 the matrix E of the system is singular if the electrical circuit contains at least one mesh consisting of branches with only ideal capacitors and voltage sources. Similarly, by Theorem 2 the matrix E is singular if the electrical circuit contains at least one node with branches with coils.□

Zeroing of the state vector

Consider descriptor linear circuit described by the equation (1) with regular pencil satisfying (2). To the electrical circuit the state-feedback

(11) $u = Kx$, $K \in \Re^{m \times n}$ (gain matrix).

is applied and the equation of closed-loop circuit has the form

(12) $E\dot{x}(t) = (A + BK)x(t)$.

We are looking for a gain matrix K such that state vector $x(t)$ of the closed-loop circuit satisfies the condition

(13) $x(t) = 0$ for $t > 0$

for any admissible initial conditions and any values of resistances, inductances and capacitances.
It will be shown that there exists a gain matrix K such that the condition (3.3) is satisfied if and only if

(14) $\mathrm{rank}\left[Es - A, B\right] = n$ for all $s \in C$.

Remark 1. The condition (14) is satisfied if and only if the matrix $\left[Es - A, B\right]$ can be reduced to the matrix $\left[0 \quad I_n\right]$ by the use of elemantary column operations [9].
Theorem 4. There exists $K \in \Re^{m \times n}$ satisfying the condition

(15) $\det\left[Es - \left(A + BK\right)\right] = \alpha \neq 0$

(α - a real number independent of s) if and only if the condition (3.4) is met.
Proof is given in [6].
The solution of the problem is based on the following theorem.
Theorem 5. There exists a gain matrix $K \in \Re^{m \times n}$ such that (13) holds if and only if the condition (14) is satisfied.
Proof. By Theorem 4 there exists K satisfying (15) if and only if the condition (14) is met. In this case, using the Laplace transform from (12) we obtain

(16) $X(s) = \left[Es - \left(A + BK\right)\right]^{-1} x_0$,

where $X(s) = \int_0^\infty x(t)e^{-st}dt$ is the Laplace transform of $x(t)$ and x_0 is the admissible initial condition.
Taking into account (15) we obtain

(17)
$$X(s) = \frac{\mathrm{Adj}\left[Es - \left(A + BK\right)\right]}{\det\left[Es - \left(A + BK\right)\right]} x_0 = \frac{\mathrm{Adj}\left[Es - \left(A + BK\right)\right]}{\alpha} x_0 = \left(P_0 + P_1 s + \ldots + P_q s^q\right) x_0$$

where $\mathrm{Adj}\left[Es - \left(A + BK\right)\right]$ denotes the adjoint matrix and $P_k \in \Re^{n \times n}$ for $k = 0, 1, ..., q$.

Applying the inverse Laplace transform to (17) we obtain

(18) $x(t) = \sum_{k=0}^{q} P_k x_0 \delta^{(k)}(t) = 0$ for $t > 0$,

where $\delta(t)$ is the Dirac impulse and $\delta^{(k)}(t)$ is k-th derivative.□

Example 3. Consider the electrical circuit shown in Fig. 3 with given resistance R, capacitances C_1, C_2 and source voltage $e = e(t)$.

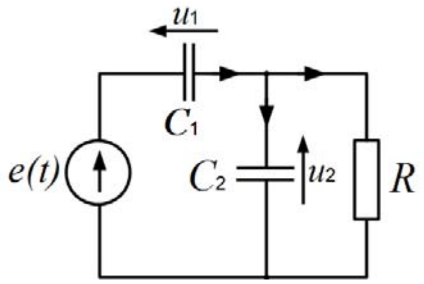

Fig. 3. Electrical circuit of Example 3.

Using Kirchhoff's laws for the electrical circuit we can write the equations

(19) $$C_1 \frac{du_1}{dt} - C_2 \frac{du_2}{dt} = \frac{u_2}{R}, \quad u_1 + u_2 = e,$$

which can be rewritten in the form

(20a) $$E \frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + Be,$$

where

(20b) $$E = \begin{bmatrix} C_1 & -C_2 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \frac{1}{R} \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The condition (14) is satisfied since

(21) $$\operatorname{rank}\left[Es - A, B\right] = \operatorname{rank}\begin{bmatrix} sC_1 & -sC_2 - \frac{1}{R} & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2$$

for all $s \in C$.

For the gain matrix $K = [k_1 \quad k_2]$ the closed-loop system matrix has the form

(22) $$\left[Es - (A + BK)\right] = \begin{bmatrix} sC_1 & -sC_2 - \frac{1}{R} \\ 1 - k_1 & 1 - k_2 \end{bmatrix}$$

and its determinant is equal to a real number $\alpha \neq 0$

(23) $$\det\left[Es - (A + BK)\right] = s\left[C_1(1 - k_2) + C_2(1 - k_1)\right] + \frac{1 - k_1}{R} = \alpha$$

for

(24) $k_1 \neq 1$ and $k_2 = \dfrac{C_1 + C_2(1 - k_1)}{C_1}$.

Therefore, for the state feedback matrix $K = [k_1 \quad k_2]$ with k_1 and k_2 defined by (3.14) we have $u_1(t) = 0$, $u_2(t) = 0$ for $t > 0$.

Example 4. Consider the electrical circuit shown in Fig. 3.2 with given resistances R_1, R_2, inductances L_1, L_2 and source current $i_s(t) = i_s$.

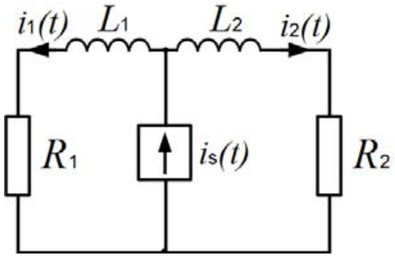

Fig. 4. Electrical circuit of Example 4.

Using Kirchhoff's laws for the electrical circuit we can write the equations

(25) $$R_1 i_1 + L_1 \frac{di_1}{dt} - R_2 i_2 - L_2 \frac{di_2}{dt} = 0, \quad i_s = i_1 + i_2,$$

which can be rewritten in the form

(26a) $$E \frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + Bi_s,$$

where

(26b) $$E = \begin{bmatrix} L_1 & -L_2 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & R_2 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The condition (14) is satisfied since

(27) $$\operatorname{rank}\left[Es - A, B\right] = \operatorname{rank}\begin{bmatrix} R_1 + sL_1 & -R_2 - sL_2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2$$

for all $s \in C$.

For the gain matrix $K = [k_1 \quad k_2]$ the closed-loop system matrix has the form

(28) $$\left[Es - (A + BK)\right] = \begin{bmatrix} R_1 + sL_1 & -R_2 - sL_2 \\ 1 - k_1 & 1 - k_2 \end{bmatrix}$$

and its determinant is equal to a real number $\alpha \neq 0$

(29)
$$\det\left[Es - (A + BK)\right] = (R_1 + sL_1)(1 - k_2) + (R_2 + sL_2)(1 - k_1) = \alpha$$

for k_1 and k_2 satisfying the equation

(30) $$\begin{bmatrix} L_2 & L_1 \\ R_2 & R_2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} L_1 + L_2 \\ (R_1 + R_2) - \alpha \end{bmatrix}.$$

The solution of (30) has the form

(31) $$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} L_2 & L_1 \\ R_2 & R_2 \end{bmatrix}^{-1}\begin{bmatrix} L_1 + L_2 \\ (R_1 + R_2) - \alpha \end{bmatrix} = \begin{bmatrix} 1 + \frac{L_1\alpha}{R_1 L_2 - R_2 L_1} \\ 1 - \frac{L_2\alpha}{R_1 L_2 - R_2 L_1} \end{bmatrix}$$

Therefore, for the state feedback $K = [k_1 \quad k_2]$ with k_1 and k_2 given by (31) we have $i_1(t) = 0$, $i_2(t) = 0$ for $t > 0$.

Remark 2. For the electrical circuit shown in Fig. 1 the condition (14) is not satisfied since

(32)

$$\operatorname{rank}\left[Es-A,B\right]=\operatorname{rank}\begin{bmatrix} sRC_1+1 & 0 & 1 & 1 & 0 \\ sC_1 & sC_2 & -sC_3 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}=2$$

for all $s \in C$.

Concluding remarks

The problem of zeroing of the state variables in descriptor electrical circuits by suitable choice of state feedbacks has been formulated and solved. It has been shown that there exists a gain matrix such that (13) holds if and only if the condition (14) is satisfied. The choice of the gain matrix of state feedbacks has been demonstrated in two examples of descriptor electrical circuits. The considerations can be extended to fractional descriptor electrical circuits [12].

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