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Probabilistic properties of sinusoidal signal autocorrelation function

***Abstract**. The paper concerns issues of probabilistic properties of the sinusoidal signal autocorrelation function. An autocorrelation function can be viewed as a random variable with fixed probability density. In the paper, results of the research on parameters of such a variable are presented. On the basis of the probability density function, the mean, the mean-square and the variance of the random variable have been determined.*

***Streszczenie**. Artykuł dotyczy problematyki probabilistycznych własności funkcji autokorelacji sygnału sinusoidalnego. Funkcja autokorelacji może być rozpatrywana jako zmienna losowa o ustalonej gęstości prawdopodobieństwa. W artykule przedstawiono wyniki badań dotyczące parametrów takiej zmiennej losowej. Na podstawie funkcji gęstości wyznaczono wartość oczekiwaną, średniokwadratową i wariancję zmiennej losowej. (**Właściwości losowe funkcji autokorelacji sygnału sinusoidalnego**)*

Keywords: autocorrelation function, mean, variance, arcsine distribution
Słowa kluczowe: funkcja autokorelacji, wartość oczekiwana, wariancja, rozkład arcsin

Introduction

In the paper, probabilistic properties of the sinusoidal signal autocorrelation function have been investigated. An autocorrelation function can be presented in the form of a random variable with a fixed probability density distribution. Based on a finite set of observations, it is possible to only estimate the random variable distribution parameters of interest. Therefore, in order to find out the real values of the random variable distribution parameters, or, in short, random variable parameters, such values are determined based on a probability density function. Random variable parameters are quantities characterizing the values which a random variable can assume. The authors have determined such parameters as the mean, the mean-square, as well as the variance of a random variable.

Sinusoidal signal autocorrelation function

The autocorrelation function of a periodic and ergodic signal $x(t)$, continuous over the time $t\in\mathbf{R}$, is defined as follows [1, 2]:

$$R_{x(t)}(\tau)=\frac{1}{T_s}\int_0^{T_s} x(t)\,x(t+\tau)\,\mathrm{d}t, \tag{1}$$

where $T_s\in\mathbf{R}_+$ is the period, $\tau\in\mathbf{R}$ is the delay of the signal $x(t)$.

Let $x(t)$ be a signal described by the formula:

$$x(t)=A_0+A\sin\left(\frac{2\pi}{T_s}t+\varphi\right), \tag{2}$$

where $A\in\mathbf{R}_+$ is the amplitude, $A_0\in\mathbf{R}$ is the permanent component, $\varphi\in\mathbf{R}$ is the initial phase of the signal $x(t)$.

On the basis of formulae (1) and (2) we obtain [1, 2] (Fig. 1):

$$R_{x(t)}(\tau)=A_0^2+\frac{A^2}{2}\cos\left(\frac{2\pi}{T_s}\tau\right). \tag{3}$$

Note that quantity (3) makes it possible to determine the mean-square of signal (2). Setting $\tau=\tau_0=0$, we obtain $R_{x(t)}(\tau_0)=A_0^2+A^2/2$.

Probabilistic properties of sinusoidal signal autocorrelation function

The autocorrelation function $R_{x(t)}(\tau)$ of signal (2) assumes values from the interval $[-A^2/2+A_0^2, A^2/2+A_0^2]$. If we assume that $R_{x(t)}(\tau)$ is a random variable, then $R_{x(t)}(\tau)$ has arcsine distribution over the given interval. The probability distribution function of the random variable $R_{x(t)}(\tau)$ has the form (Fig. 1):

$$f_{R_{x(t)}(\tau)}(r)=\begin{cases}\dfrac{1}{\pi\sqrt{\dfrac{A^4}{4}-\left(r-A_0^2\right)^2}}, & \left|r-A_0^2\right|<\dfrac{A^2}{2},\\[2ex] 0, & \left|r-A_0^2\right|\ge\dfrac{A^2}{2}.\end{cases} \tag{4}$$

The cumulative distribution function of the random variable $R_{x(t)}(\tau)$ is determined based on the formula:

$$F_{R_{x(t)}(\tau)}(r)=\int_{-\infty}^{r} f_{R_{x(t)}(\tau)}(u)\,\mathrm{d}u =\begin{cases}\dfrac{2}{\pi}\arcsin\left(\sqrt{\dfrac{r-A_0^2}{A^2}+\dfrac{1}{2}}\right), & \left|r-A_0^2\right|\le\dfrac{A^2}{2},\\[2ex] 0, & \left|r-A_0^2\right|>\dfrac{A^2}{2},\end{cases} \tag{5}$$

whereas the inverse cumulative distribution function of the random variable $R_{x(t)}(\tau)$ according to the relationship (Fig. 2):

$$F^{-1}{}_{R_{x(t)}(\tau)}(r)=\begin{cases}A_0^2-\dfrac{A^2}{2}\cos(\pi r), & 0\le r\le 1,\\ 0, & r<0\ \vee\ r>1.\end{cases} \tag{6}$$

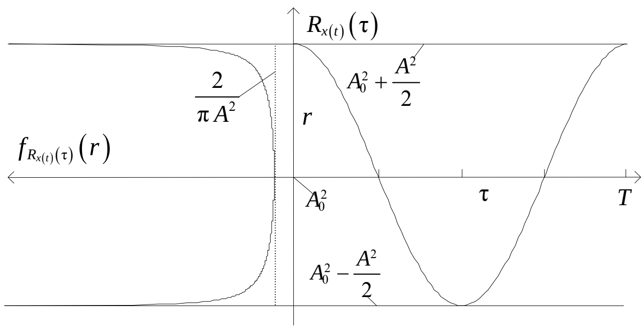

Fig.1. An the autocorrelation function $R_{x(t)}(\tau)$ and the probability density function $f_{R_{x(t)}(\tau)}(r)$

In practice, a distribution function is a more effective tool to investigate probability than a density function. This is the case because simpler mathematical apparatus is required to describe a distribution function. An inverse distribution function is often determined in order to design, by means of a method of uniform distribution transformation, a random number generator with a specific probability distribution [3].

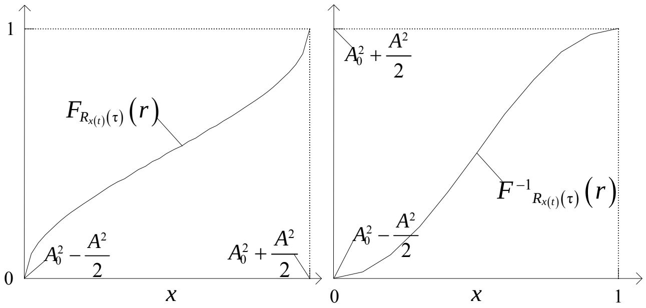

Fig.2. The cumulative distribution function $F_{Rx(t)(\tau)}(r)$ and the inverse cumulative distribution function $F^{-1}{}_{Rx(t)(\tau)}(r)$

On the basis of formula (4) we can determine the mean of the random variable $R_{x(t)}(\tau)$:

$$E\left[R_{x(t)}(\tau)\right] = \int_{A_0^2-\frac{A^2}{2}}^{A_0^2+\frac{A^2}{2}} \xi\, f_{R_{x(t)}(\tau)}(\xi)\,\mathrm{d}\xi = A_0^2. \tag{7}$$

Similarly, we determine the mean-square of the random variable $R_{x(t)}(\tau)$:

$$E\left[\left(R_{x(t)}(\tau)\right)^2\right] = \int_{A_0^2-\frac{A^2}{2}}^{A_0^2+\frac{A^2}{2}} \xi^2\, f_{R_{x(t)}(\tau)}(\xi)\,\mathrm{d}\xi = A_0^4 + \frac{1}{8}A^4. \tag{8}$$

The variance of the random variable $R_{x(t)}(\tau)$ is determined based on the formula:

$$\begin{aligned} Var\left[R_{x(t)}(\tau)\right] &= \int_{A_0^2-\frac{A^2}{2}}^{A_0^2+\frac{A^2}{2}} \left(\xi - E\left[R_{x(t)}(\tau)\right]\right)^2 f_{R_{x(t)}(\tau)}(\xi)\,\mathrm{d}\xi \\ &= E\left[\left(R_{x(t)}(\tau)\right)^2\right] - E^2\left[R_{x(t)}(\tau)\right] = \frac{1}{8}A^4. \end{aligned} \tag{9}$$

Quantities (7) and (8), respectively, are ordinary moments of the first and the second order, whereas quantity (9) is the second central moment [3].

Analogously, we can determine a formula for calculating an ordinary kth, $k \in \mathbf{N}\backslash\{0\}$, order moment of the random variable $R_{x(t)}(\tau)$. If $A < \sqrt{2}\,|A_0|$, then:

$$\begin{aligned} E\left[\left(R_{x(t)}(\tau)\right)^k\right] &= \int_{A_0^2-\frac{A^2}{2}}^{A_0^2+\frac{A^2}{2}} \xi^k f_{R_{x(t)}(\tau)}(\xi)\,\mathrm{d}\xi \\ &= A_0^{2k}\,{}_2F_1\left(\frac{1-k}{2}, -\frac{k}{2}, 1, \frac{A^4}{4A_0^4}\right), \end{aligned} \tag{10}$$

where ${}_2F_1(\cdot)$ is a hypergeometrical function [4]. The form of formula (10) has been obtained by means of the computer program Mathematica [5].

Since formula (10) allows to calculate a kth order moment of the random variable only to a limited extent, the authors, making use of the results presented in the works [6, 7], have derived the formula:

$$\begin{aligned} &E\left[R_{x(t)}(\tau)\right] = A_0^2, \\ &E\left[\left(R_{x(t)}(\tau)\right)^2\right] = A_0^4 + \frac{1}{8}A^4, \\ &E\left[\left(R_{x(t)}(\tau)\right)^3\right] = A_0^6 + \frac{3}{8}A^4 A_0^2, \\ &E\left[\left(R_{x(t)}(\tau)\right)^4\right] = A_0^8 + \frac{6}{8}A^4 A_0^4 + \frac{3}{128}A^8, \\ &\vdots \\ &E\left[\left(R_{x(t)}(\tau)\right)^k\right] = A_0^{2k} + \sum_{i=1}^{k} \binom{k}{i} \binom{\frac{1}{2}(i-1)\left(1+(-1)^i\right)}{\frac{1}{2}\left(\frac{i}{2}-1\right)\left(1+(-1)^i\right)} A^{2i} A_0^{2(k-i)} \frac{1+(-1)^i}{4^i}, \end{aligned} \tag{11}$$

which makes it possible to determine a kth order moment without setting initial conditions concerning the parameters of signal (2).

Verification of the theoretical results

The results obtained above allow us to estimate the information content of the autocorrelogram which is a graphical representation of the autocorrelation function. This information concerns, among others, the energy-related parameters in time domain (the mean value, the mean square, the root-mean square) and in the frequency domain (power spectrum) [1].

Let us consider the following three measurement setups in which the frequency f_s of the signal $x(t)$ depends on the trigger frequency f_w in the following ways:

$$\begin{aligned} f_w &= f_s \rightarrow T_w = T_s, \\ f_w &> f_s \rightarrow T_w < T_s, \\ f_w &< f_s \rightarrow T_w > T_s, \end{aligned} \tag{12}$$

where T_w is the trigger period.

Let us assume that the signal $x(t)$ has been sampled with the frequency f_p=$M\,f_w$ and the initial phase φ=0 so that:

$$\mathrm{x}[i] = x(iT_p) = A_0 + A\sin\left(2\pi i \frac{1}{M}\frac{f_s}{f_w}\right),\; i = 0,1,\ldots,2M-1, \tag{13}$$

where T_p is the sampling period, M is the number of samples per T_s.

In each situation the following estimator of the $R_{x(t)}(\tau)$ function has been constructed [1], [2]:

$$R_{\mathrm{x}[i]}[k] = \frac{1}{M}\sum_{i=0}^{M-1} \mathrm{x}[i]\,\mathrm{x}[i+k],\quad k = 0, 1, \ldots, M-1, \tag{14}$$

and the parameter estimation error has been evaluated as (7)-(9):

$$\delta_\theta = \left|\frac{\tilde{\theta}-\theta}{\theta}\right| 100\ \%, \tag{15}$$

where θ is the true value of the parameter, $\tilde{\theta}$ is the value of the estimator of the parameter, evaluated with the use of samples of the autocorrelogram. The estimators have been calculated with formulas which are widely known from the literature [1]-[4].

Measurement setup 1: f_w=f_s.

Figure 3 and table 1 are example illustrations of the first measurement setup.

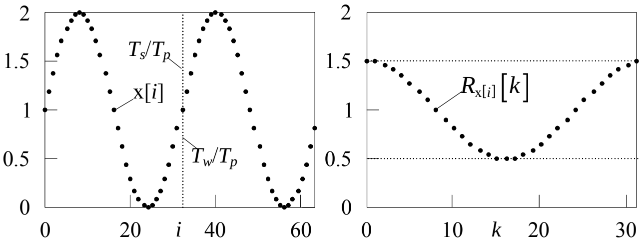

Fig.3. Samples of the signal and of the autocorrelogram (M=32, f_s=f_w=100 Hz, A=A_0=1 V)

Table 1. Results of the parameters estimation (7)-(9)

	Mean	Mean-square	Variance
θ	1 V	1.125 V^2	0.125 V^2
$\tilde{\theta}$	1 V	1.125 V^2	0.125 V^2
δ_θ	0	0	0

It can be easily noticed that in the discussed case the autocorrelogram contains all the information about the signal $x(t)$.

Measurement setup 2: f_w<f_s.

Figure 4 and table 2 are example illustrations of the second measurement setup.

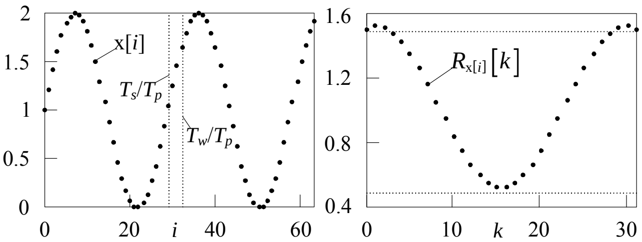

Fig.4. Samples of the signal and of the autocorrelogram (M=32, f_s=100 Hz, f_w= 90 Hz, A=A_0=1 V)

Table 2. Results of the parameters estimation (7)-(9)

	Mean	Mean-square	Variance
θ	1 V	1.125 V^2	0.125 V^2
$\tilde{\theta}$	1.072 V	1.281 V^2	0.132 V^2
δ_θ	7.3 %	14 %	5.7 %

Because the initial phase in the second period of the signal $x(t)$ is different from zero, the averaging process leads to the loss of a part of the information about signal $x(t)$ contained in the autocorrelogram.

Measurement setup 3: f_w>f_s.

Figure 5 and table 3 are example illustrations of the third measurement setup.

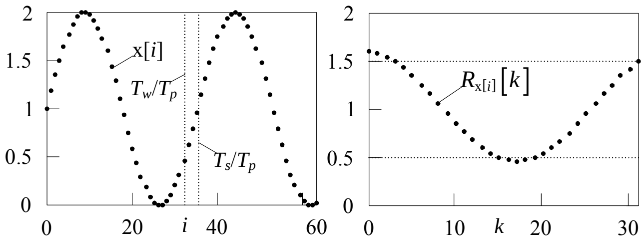

Fig.5. Samples of the signal and of the autocorrelogram (M=32, f_s=100 Hz, f_w= 110 Hz, A=A_0=1 V)

Table 3. Results of the parameters estimation (7)-(9)

	Mean	Mean-square	Variance
θ	1 V	1.125 V^2	0.125 V^2
$\tilde{\theta}$	0.982 V	1.112 V^2	0.148 V^2
δ_θ	1.9 %	1.2 %	19 %

Just like in the second measurement setup, the change of phase in the second period of the signal $x(t)$ results the loss of a part of the information about signal $x(t)$ contained in the autocorrelogram.

In realistic measurement setups the initial phase in successive periods of the signal undergoes some fluctuations. The relations obtained above may be used to verify the precision of the evaluated estimators with the use of real autocorrelogram samples.

Conclusion

In the paper, results concerning research on the probabilistic properties of the sinusoidal signal autocorrelation function have been presented. The conducted research comprised determining the parameters of a random variable which assumes the autocorrelation function values. The mean, the mean-square and the variance of a random variable have been determined. In practice, such quantities are estimated based on signal or autocorrelogram samples. The obtained formulae make it possible to determine actual values of the distribution parameters of the autocorrelation function values.

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