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Determining autocorrelation function values from six sinusoidal signal samples

***Abstract**. In the paper, it is shown that at a given moment of time the actual values of the sinusoidal signal autocorrelation function can be determined in an unambiguous way on the basis of three samples of the signal and three samples of its time-shifted copy. Based on this, an algorithm making it possible to determine an autocorrelogram has been devised. The employment of the devised algorithm substantially reduces the time consumption of determining an autocorrelogram.*

***Streszczenie**. W artykule pokazano, że w ustalonej chwili czasowej rzeczywiste wartość funkcji autokorelacji sygnału sinusoidalnego można wyznaczyć w sposób jednoznaczny na podstawie trzech próbek sygnału i trzech próbek jego własnej, przesuniętej w czasie kopii. Na tej podstawie opracowano algorytm umożliwiający wyznaczanie autokorelogramu. (**Wyznaczanie funkcji autokorelacji na podstawie sześciu sygnałów sinusoidalnych**)*

Keywords: autocorrelation function, autocorrelogram, time complexity, time consumption
Słowa kluczowe: funkcja autokorelacji, autokorelogram, złożoność obliczeniowa, czasochłonność

Introduction

The autocorrelation function is an established tool to analyze signal properties and to compare signals. In many practical applications, the autocorrelation function is used for checking to what degree the value of a signal at a fixed moment of time influences the value of the signal at a certain moment in the future. Such investigation consists in comparing a signal with its time-shifted copy. In practice, actual values of the signal autocorrelation function are determined or estimated from samples of the signal. Based on this, an autocorrelogram is prepared which is a graphical and numerical representation of the autocorrelation function. An autocorrelogram is determined in order to investigate the autocorrelation function properties. Because of the quadratic time complexity of the autocorrelation algorithm, determining an autocorrelogram is time-consuming.

In the paper, it is shown that at a fixed moment of time, actual values of the sinusoidal signal autocorrelation function can be unambiguously determined based on three samples of the signal and three samples of its time-shifted copy. Then determining an autocorrelogram can be carried out by means of an algorithm of linear time complexity. This results in significant reduction in the time consumption of determining an autocorrelogram.

Sinusoidal signal autocorrelation function

The autocorrelation function of a continuous in the time $t\in\mathbf{R}$ periodic and ergodic signal $x(t)$ is defined as follows [1-4]:

$$R_{x(t)}(\tau)=\frac{1}{T}\int_0^T x(t)\,x(t+\tau)\,\mathrm{d}t, \tag{1}$$

where $T\in\mathbf{R}_+$ is the period, $\tau\in\mathbf{R}$ is the delay of $x(t)$.

Let $x(t)$ be a sinusoidal signal described by the formula:

$$x(t)=A_0+A\sin\left(\frac{2\pi}{T}t+\varphi\right), \tag{2}$$

where $A\in\mathbf{R}_+$ is the amplitude, $A_0\in\mathbf{R}$ is the constant component, $\varphi\in\mathbf{R}$ is the initial phase of $x(t)$.

Based on formulae (1) and (2), we obtain [1-4]:

$$\begin{aligned} R_{x(t)}(\tau)&=\frac{1}{T}\int_0^T x(t)\,x(t+\tau)\,\mathrm{d}t= \\ &=\frac{1}{T}\int_0^T\left(A_0+A\sin\left(\frac{2\pi}{T}t+\varphi\right)\right)\left(A_0+A\sin\left(\frac{2\pi}{T}(t+\tau)+\varphi\right)\right)\mathrm{d}t \\ &=A_0^2+\frac{A^2}{2}\cos\left(\frac{2\pi}{T}\tau\right). \end{aligned} \tag{3}$$

In practice, signal (2) undergoes sampling, i.e. discretization in the time domain. Let us consider the classic approach and assume that the discretization is uniform, and that the sampling theorem is fulfilled. Then, based on M samples (per period):

$$\mathrm{x}[i]=x\left(\frac{T}{M}i\right)=A_0+A\sin\left(\frac{2\pi}{M}i+\varphi\right),\ i=0,1,\dots,M-1, \tag{4}$$

$$\mathrm{x}[i+k]=x\left(\frac{T}{M}(i+k)\right)=A_0+A\sin\left(\frac{2\pi}{M}(i+k)+\varphi\right),\quad k=0,1,\dots,M-1, \tag{5}$$

of signal (2), we obtain the quantity:

$$\begin{aligned} R_{\mathrm{x}[i]}[k]&=\frac{1}{M}\sum_{i=0}^{M-1}\mathrm{x}[i]\,\mathrm{x}[i+k]=\frac{1}{M}\sum_{i=0}^{M-1}x\left(\frac{T}{M}i\right)x\left(\frac{T}{M}(i+k)\right) \\ &=\frac{1}{M}\sum_{i=0}^{M-1}\left(A_0+A\sin\left(\frac{2\pi}{M}i+\varphi\right)\right)\left(A_0+A\sin\left(\frac{2\pi}{M}(i+k)+\varphi\right)\right), \end{aligned} \tag{6}$$

on whose basis the value of function (3) can be estimated.

Quantity (6) constitutes a mathematical description of an iterative algorithm making it possible to determine an autocorrelogram. Such an algorithm can be presented in the following way:

***Step* 1**: Fix the values of the parameters of signal (2), the values of the signal processing parameters, and the initial value k_0 of the delay k.
***Step* 2**: Using formulae (4) and (5), determine and store samples of the signal and samples of its time-shifted copy.
***Step* 3**: Determine and store the value of quantity (6).
***Step* 4**: Repeat steps 2 and 3 for subsequent values k_0 of the delay k in order to obtain a correlogram.

It can be noted that the number of operations required to determine an autocorrelogram is equal to M^2. Therefore, the time complexity of the algorithm is quadratic.

In order to compare quantities (3) and (6) one has to determine the values of these quantities at the same fixed moment of time. It requires finding a relationship between the delays k and τ. Let us assume that for any fixed value τ_0 of the delay τ, there exists a value:

$$k_0 = \frac{M}{T}\tau_0, \tag{7}$$

of the delay k, such that $k_0 \in \mathbf{N}$. Then the values of quantities (3) and (6) are determined at the same moment of time.

Determining autocorrelation function values from six sinusoidal signal samples

First of all, let us show that for M=2 initial samples (per period) of signal (2) and for M=2 initial samples (per period) of its time-shifted copy, we obtain $R_{x[i]}[k_0] \neq R_{x(t)}(\tau_0)$.

Let us assume M=2, thus, based on formula (6), we obtain:

$$\begin{aligned} R_{x[i]}[k_0] &= \frac{1}{2}\sum_{i=0}^{1} x[i]\,x[i+k_0] \\ &= \frac{1}{2}x[0]\,x[k_0] + \frac{1}{2}x[1]\,x[k_0+1] \\ &= \frac{1}{2}x(0)\,x\left(\frac{T}{2}k_0\right) + \frac{1}{2}x\left(\frac{T}{2}\right)x\left(\frac{T}{2}(k_0+1)\right), \end{aligned} \tag{8}$$

where:

$$\begin{aligned} x(0) &= A_0 + A\sin(\varphi), \\ x\left(\frac{T}{2}\right) &= A_0 + A\sin(\pi+\varphi), \\ x\left(\frac{T}{2}k_0\right) &= A_0 + A\sin(\pi k_0+\varphi), \\ x\left(\frac{T}{2}(k_0+1)\right) &= A_0 + A\sin(\pi(k_0+1)+\varphi). \end{aligned} \tag{9}$$

Let us note that k_0 assumes the values 0 and 1, while τ_0 the values 0 and $T/2$. Then the values of quantities (3) and (8) are determined at the same moments of time.

Since:

$$\begin{aligned} \sin(\pi+\varphi) &= -\sin(\varphi), \\ \sin(\pi k_0+\varphi) &= (-1)^{k_0}\sin(\varphi), \\ \sin(\pi(k_0+1)+\varphi) &= (-1)^{k_0+1}\sin(\varphi), \end{aligned} \tag{10}$$

then:

$$R_{x[i]}[k_0] = A_0^2 + (-1)^{k_0} A^2\sin^2(\varphi). \tag{11}$$

Putting $k_0 = 2\tau_0/T$ in formula (11), we obtain:

$$\begin{aligned} R_{x[i]}[k_0] &= R_{x[i]}\left[\frac{2}{T}\tau_0\right] \\ &= A_0^2 + (-1)^{\frac{2}{T}\tau_0} A^2\sin^2(\varphi). \end{aligned} \tag{12}$$

Comparing formulae (3) and (12) we note instantly that $R_{x[i]}[k_0] \neq R_{x(t)}(\tau_0)$. This means that based on M=2 initial samples of signal (2) and M=2 initial samples of its time-shifted copy, it is impossible to unambiguously determine the actual value of function (3).

Let us now formulate the following theorem:

Theorem 1

If for any fixed value τ_0 of the delay τ, there exists a value:

$$k_0 = \frac{3}{T}\tau_0, \tag{13}$$

of the delay k, such that $k_0 \in \mathbf{N}$, then for M=3 initial samples of signal (2) and for M=3 initial samples of its time-shifted copy, we obtain $R_{x[i]}[k_0] = R_{x(t)}(\tau_0)$.

Proof. Let us assume M=3, thus on the basis of formula (6) we obtain:

$$\begin{aligned} R_{x[i]}[k_0] &= \frac{1}{3}\sum_{i=0}^{2} x[i]\,x[i+k_0] \\ &= \frac{1}{3}x[0]\,x[k_0] + \frac{1}{3}x[1]\,x[k_0+1] + \frac{1}{3}x[2]\,x[k_0+2] \\ &= \frac{1}{3}x(0)\,x\left(\frac{T}{3}k_0\right) + \frac{1}{3}x\left(\frac{T}{3}\right)x\left(\frac{T}{3}(k_0+1)\right) \\ &\quad + \frac{1}{3}x\left(\frac{2T}{3}\right)x\left(\frac{T}{3}(k_0+2)\right), \end{aligned} \tag{14}$$

where:

$$\begin{aligned} x(0) &= A_0 + A\sin(\varphi), \\ x\left(\frac{T}{3}\right) &= A_0 + A\sin\left(\frac{2\pi}{3}+\varphi\right), \\ x\left(\frac{2T}{3}\right) &= A_0 + A\sin\left(\frac{4\pi}{3}+\varphi\right), \\ x\left(\frac{T}{3}k_0\right) &= A_0 + A\sin\left(\frac{2\pi}{3}k_0+\varphi\right), \\ x\left(\frac{T}{3}(k_0+1)\right) &= A_0 + A\sin\left(\frac{2\pi}{3}(k_0+1)+\varphi\right), \\ x\left(\frac{T}{3}(k_0+2)\right) &= A_0 + A\sin\left(\frac{2\pi}{3}(k_0+2)+\varphi\right). \end{aligned} \tag{15}$$

Let us note that k_0 assumes the values 0, 1, and 2, while τ_0 the values 0, $T/3$, and $2T/3$. Then the values of quantities (3) and (14) are determined at the same moments of time.

Since:

$$\begin{aligned} \sin\left(\frac{2\pi}{3}+\varphi\right) &= \frac{\sqrt{3}}{2}\cos(\varphi) - \frac{1}{2}\sin(\varphi), \\ \sin\left(\frac{4\pi}{3}+\varphi\right) &= -\frac{\sqrt{3}}{2}\cos(\varphi) - \frac{1}{2}\sin(\varphi), \\ \sin\left(\frac{2\pi}{3}k_0+\varphi\right) &= \sin\left(\frac{2\pi}{3}k_0\right)\cos(\varphi) + \cos\left(\frac{2\pi}{3}k_0\right)\sin(\varphi), \\ \sin\left(\frac{2\pi}{3}(k_0+1)+\varphi\right) &= \left(\frac{\sqrt{3}}{2}\cos\left(\frac{2\pi}{3}k_0\right) - \frac{1}{2}\sin\left(\frac{2\pi}{3}k_0\right)\right)\cos(\varphi) \\ &\quad - \left(\frac{\sqrt{3}}{2}\sin\left(\frac{2\pi}{3}k_0\right) + \frac{1}{2}\cos\left(\frac{2\pi}{3}k_0\right)\right)\sin(\varphi), \\ \sin\left(\frac{2\pi}{3}(k_0+2)+\varphi\right) &= \left(\frac{\sqrt{3}}{2}\sin\left(\frac{2\pi}{3}k_0\right) - \frac{1}{2}\cos\left(\frac{2\pi}{3}k_0\right)\right)\sin(\varphi) \\ &\quad - \left(\frac{1}{2}\sin\left(\frac{2\pi}{3}k_0\right) + \frac{\sqrt{3}}{2}\cos\left(\frac{2\pi}{3}k_0\right)\right)\cos(\varphi), \end{aligned} \tag{16}$$

then:

$$R_{x[i]}[k_0] = A_0^2 + \frac{A^2}{2}\cos^2(\varphi)\cos\left(\frac{2\pi}{3}k_0\right) + \frac{A^2}{2}\sin^2(\varphi)\cos\left(\frac{2\pi}{3}k_0\right). \tag{17}$$

Ordering and reducing formula (17), we obtain:

$$R_{x[i]}[k_0] = A_0^2 + \frac{A^2}{2}\cos\left(\frac{2\pi}{3}k_0\right). \tag{18}$$

Putting $k_0=3\tau_0/T$ in formula (18), we obtain:

$$R_{x[i]}[k_0] = R_{x[i]}\left[\frac{3}{T}\tau_0\right] = A_0^2 + \frac{A^2}{2}\cos\left(\frac{2\pi}{T}\tau_0\right) = R_{x(t)}(\tau_0). \tag{19}$$

It has been proven that to unambiguously determine the actual value of the autocorrelation function $R_{x(t)}(\tau)$ of signal (2), it is sufficient to have $M=3$ initial samples of the signal and $M=3$ initial samples of its time-shifted copy.

The following theorem is also true:

Theorem 2

If for any fixed value τ_0 of the delay τ, there exists a value $k_0=3\tau_0/T$ of the delay k, such that $k_0\in\mathbf{N}$, then for $M=3$ subsequent samples of signal (2) and for $M=3$ subsequent samples of its time-shifted copy, we obtain $R_{x[i]}[k_0]=R_{x(t)}(\tau_0)$.

We leave the proof of *Theorem* 2 to the reader. In the proof, it should be noted that for $M=3$ and $n\in\mathbf{N}$:

$$\begin{aligned} R_{x[i]}[k_0] &= \frac{1}{3}\sum_{i=n}^{n+2} x[i]\,x[i+k_0] \\ &= \frac{1}{3}x[n]\,x[k_0+n] + \frac{1}{3}x[n+1]\,x[k_0+n+1] + \frac{1}{3}x[n+2]\,x[k_0+n+2] \\ &= \frac{1}{3}x\left(\frac{T}{3}n\right)x\left(\frac{T}{3}(k_0+n)\right) + \frac{1}{3}x\left(\frac{T}{3}(n+1)\right)x\left(\frac{T}{3}(k_0+n+1)\right) \\ &\quad + \frac{1}{3}x\left(\frac{T}{3}(n+2)\right)x\left(\frac{T}{3}(k_0+n+2)\right), \end{aligned} \tag{20}$$

where n is the index of the sample from which determining the values of quantity (20) begins.

It should be borne in mind that if we abandon the physical interpretation understood in terms of signal sampling, and we take into account τ_0, for which a certain $l_0=3\tau_0/T$ exists which is not the index of a sample, but a real number, then the formula:

$$R_{x(t)}(l_0) = \frac{1}{3}x(0)\,x\left(\frac{T}{3}l_0\right) + \frac{1}{3}x\left(\frac{T}{3}\right)x\left(\frac{T}{3}(l_0+1)\right) + \frac{1}{3}x\left(\frac{2T}{3}\right)x\left(\frac{T}{3}(l_0+2)\right), \tag{21}$$

makes it possible to determine the actual value of the autocorrelation function $R_{x(t)}(\tau_0)$ of signal (2). The explanation of this fact (it also refers to *Theorem* 2) follows from the stationariness of signal (2), i.e. the mean and the autocorrelation function of the signal do not depend on the initial moment of time at which they are determined [3].

The above remark will be helpful in answering the following question. In the situation when we a priori assume $M>3$, can *Theorem* 1 be used to determine an autocorrelogram adequate to the one obtained by means of formula (6)? The answer is positive. It will suffice to apply the formula:

$$R_{x[i]}[k] = \frac{1}{3}x(0)\,x\left(T\frac{k}{M}\right) + \frac{1}{3}x\left(\frac{T}{3}\right)x\left(T\left(\frac{k}{M}+\frac{1}{3}\right)\right) + \frac{1}{3}x\left(\frac{2T}{3}\right)x\left(T\left(\frac{k}{M}+\frac{2}{3}\right)\right), \tag{22}$$

which follows from relation (14).

Quantity (22) constitutes a mathematical description of an iterative algorithm making it possible to determine an autocorrelogram. It follows from formula (22) that the number of the significant operations necessary to determine an autocorrelogram is equal to $3M$. Thus, the algorithm time complexity is linear. Since $M^2/3M=M/3$, then in comparison with the algorithm described by formula (6), $M/3$-fold reduction in the time consumption of the autocorrelogram determination will take place.

Conclusion

It has been shown in the paper that actual values of the autocorrelation function of a sinusoidal signal can be determined based on six samples of the signal. It has also been shown that the obtained results can be applied to designing an algorithm that makes it possible to reduce the time consumption of determining an autocorrelogram without losing information about the autocorrelation function.

The authors have presented the results concerning the sinusoidal signal autocorrelation function, bearing in mind that such a function is known, and its actual values can be determined from an integral formula. In such a case, determining actual values of the autocorrelation function by means of samples may be open to doubt, even if we take into account the fact that there are as many as six samples and they are easy to obtain. However, it should also be taken into consideration that in contemporary measurement systems, processing operations of a signal are almost exclusively performed on its digital representation. Searching for ways of reducing the time consumption of such operations is therefore justified.

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