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On certain transformation properties of Maxwell's equations

Streszczenie:** W prezentowanej pracy analizowane jest zagadnienie przekształcenia równań Maxwella w próżni dla pewnej klasy transformacji współrzędnych czasoprzestrzennych, które mogą w szczególności opisywać przejście między dwoma układami odniesienia poruszającymi się względem siebie ruchem prostoliniowym z dowolnie zmieniającą się prędkością. **Przekształcenia równań Maxwella w próżni dla pewnej klasy transformacji współrzędnych czasoprzestrzennych

***Abstract:** In the presented paper a transformation problem for Maxwell's equations in a vacuum is analysed within a certain transformation class of space-time coordinates. They may, in particular, to describe a transition between two reference frames moving with respect to one another in a single direction at arbitrary varying speed.*

Słowa kluczowe: równania Maxwella, transformacje współrzędnych czasoprzestrzennych, współrzędne harmoniczne
Keywords: Maxwell's equations, transformation class of space-time coordinates, harmonic coordinates

Introduction

In the presented paper a transformation problem for Maxwell's equations in a vacuum is analysed within a certain transformation class of space-time coordinates. The transformations under consideration may, in particular, describe a transition between two reference frames moving with respect to one another in a single direction at arbitrary varying speed. Making use of the obtained transformation dependencies the formulas for a front electromagnetic wave velocity were derived for a system remaining in a linear non-uniform motion with respect to an arbitrary system. The presented work continues our research reported elsewhere [5], [6].

Transformation class under analysis

We have considered the class of transformed space-time coordinates defined by the relations

$$x=\varphi_1(x',t'),\quad y=y',\quad z=z',\quad t=\varphi_2(x',t') \tag{1}$$

where x, y, z and t designate Cartesian coordinates and time, respectively, in an arbitrary inertial reference frame, called hereafter 'resting'. It is assumed that φ_1 i φ_2 functions are invertible, i.e. the Jacobian for the transformation (1) is non-zero, and 2 times differentiable. An inverse transformation to (1) can be represented as:

$$x'=\xi_1(x,t),\quad y'=y,\quad z'=z,\quad t'=\xi_2(x,t) \tag{2}$$

Formulas (1), (2) can thus in particular describe transformation relations between two physical reference frames, i.e. the resting one and the one in a linear motion relative to the former along the OX axis at an arbitrary changing velocity. However, it should be noted that the primed quantities occurring in the general formulation, not necessarily shall have any direct physical or geometrical interpretation.

Differential operators

In order to represent Maxwell's equations in the primed coordinates it is necessary to express the operation of differentiation with respect to x, y, z, t as dependent on x', y', z', t'. Based on (2) these operations can be written as:

$$\begin{gathered}\frac{\partial}{\partial x}=\frac{\partial}{\partial x'}\frac{\partial\xi_1}{\partial x}+\frac{\partial}{\partial t'}\frac{\partial\xi_2}{\partial x},\quad \frac{\partial}{\partial y}=\frac{\partial}{\partial y'},\quad \frac{\partial}{\partial z}=\frac{\partial}{\partial z'},\\ \frac{\partial}{\partial t}=\frac{\partial}{\partial x'}\frac{\partial\xi_1}{\partial t}+\frac{\partial}{\partial t'}\frac{\partial\xi_2}{\partial t}\end{gathered} \tag{3}$$

To make the right sides (3) dependent solely on the primed variables we shall express the derivatives of the functions ξ_1 and ξ_2 as variables of x' and t'. Thus we apply operators (3) to functions φ_1 and φ_2:

$$\begin{aligned}&\frac{\partial\varphi_1}{\partial x'}\frac{\partial\xi_1}{\partial x}+\frac{\partial\varphi_1}{\partial t'}\frac{\partial\xi_2}{\partial x}=1\\ &\frac{\partial\varphi_1}{\partial x'}\frac{\partial\xi_1}{\partial t}+\frac{\partial\varphi_1}{\partial t'}\frac{\partial\xi_2}{\partial t}=0\\ &\frac{\partial\varphi_2}{\partial x'}\frac{\partial\xi_1}{\partial x}+\frac{\partial\varphi_2}{\partial t'}\frac{\partial\xi_2}{\partial x}=0\\ &\frac{\partial\varphi_2}{\partial x'}\frac{\partial\xi_1}{\partial t}+\frac{\partial\varphi_2}{\partial t'}\frac{\partial\xi_2}{\partial t}=1\end{aligned} \tag{4}$$

From these equations we obtain:

$$\begin{aligned}&\frac{\partial\xi_1}{\partial x}=\frac{1}{J}\frac{\partial\varphi_2}{\partial t'}\qquad \frac{\partial\xi_2}{\partial x}=-\frac{1}{J}\frac{\partial\varphi_2}{\partial x'}\\ &\frac{\partial\xi_1}{\partial t}=-\frac{1}{J}\frac{\partial\varphi_1}{\partial t'}\qquad \frac{\partial\xi_2}{\partial x}=\frac{1}{J}\frac{\partial\varphi_1}{\partial x'}\end{aligned} \tag{5}$$

where:

$$J=\frac{\partial\varphi_1}{\partial x'}\frac{\partial\varphi_2}{\partial t'}-\frac{\partial\varphi_1}{\partial t'}\frac{\partial\varphi_2}{\partial x'} \tag{6}$$

is the Jacobian of the transformation (1). Substituting (5) to (3) we finally arrive at:

$$\begin{aligned}&\frac{\partial}{\partial x}=\frac{1}{J}\left(\frac{\partial\varphi_2}{\partial t'}\frac{\partial}{\partial x'}-\frac{\partial\varphi_2}{\partial x'}\frac{\partial}{\partial t'}\right)\\ &\frac{\partial}{\partial t}=\frac{1}{J}\left(\frac{\partial\varphi_1}{\partial x'}\frac{\partial}{\partial t'}-\frac{\partial\varphi_1}{\partial t'}\frac{\partial}{\partial x'}\right)\end{aligned} \tag{7}$$

Maxwell's equations in the primed coordinates

In the non-primed reference frame Maxwell's equations take the form:

$$\begin{aligned}&\operatorname{rot}\vec{B}=\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t},\quad \operatorname{rot}\vec{E}=-\frac{\partial\vec{B}}{\partial t},\\ &\operatorname{div}\vec{B}=0\qquad \operatorname{div}\vec{E}=0\end{aligned} \tag{8}$$

where: $\boldsymbol{B}(B_x,\ B_y,\ B_z)$ – denote magnetic induction and $\boldsymbol{E}(E_x, E_y, E_z)$ the electric field intensity.

Developing the first two equations explicitly (8) and applying operators (7) to them, following a few more transformations we obtain:

(9)
$$J\left(\frac{\partial B_3}{\partial y}-\frac{\partial B_2}{\partial z}\right)=\frac{1}{c^2}\left(\frac{\partial E_1}{\partial t'}\frac{\partial \varphi_1}{\partial x'}-\frac{\partial E_1}{\partial x'}\frac{\partial \varphi_1}{\partial t'}\right)$$
$$J\frac{\partial B_1}{\partial z}-\frac{\partial}{\partial x'}\left[B_3\frac{\partial \varphi_2}{\partial t'}-\frac{1}{c^2}E_2\frac{\partial \varphi_1}{\partial t'}\right]=\frac{1}{c^2}\frac{\partial}{\partial t'}\left(E_2\frac{\partial \varphi_1}{\partial x'}-c^2 B_3\frac{\partial \varphi_2}{\partial x'}\right)$$
$$\frac{\partial}{\partial x'}\left[B_2\frac{\partial \varphi_2}{\partial t'}+\frac{1}{c^2}E_3\frac{\partial \varphi_1}{\partial t'}\right]-J\frac{\partial B_1}{\partial y}=\frac{1}{c^2}\frac{\partial}{\partial t'}\left(E_3\frac{\partial \varphi_1}{\partial x'}+c^2\frac{\partial \varphi_2}{\partial x'}\right)$$

(10)
$$J\left(\frac{\partial E_3}{\partial y}-\frac{\partial E_2}{\partial z}\right)=-\left(\frac{\partial B_1}{\partial t'}\frac{\partial \varphi_1}{\partial x'}-\frac{\partial B_1}{\partial x'}\frac{\partial \varphi_1}{\partial t'}\right)$$
$$J\frac{\partial E_1}{\partial z}-\frac{\partial}{\partial x'}\left[E_3\frac{\partial \varphi_2}{\partial t'}+B_2\frac{\partial \varphi_1}{\partial t'}\right]=-\frac{\partial}{\partial t'}\left(B_2\frac{\partial \varphi_1}{\partial x'}+E_3\frac{\partial \varphi_2}{\partial x'}\right)$$
$$\frac{\partial}{\partial x'}\left[E_2\frac{\partial \varphi_2}{\partial t'}-B_3\frac{\partial \varphi_1}{\partial t'}\right]-J\frac{\partial E_1}{\partial y}=-\frac{\partial}{\partial t'}\left(B_3\frac{\partial \varphi_1}{\partial x'}-E_2\frac{\partial \varphi_2}{\partial x'}\right)$$

Assuming that the primed variables occurring in (9) and (10) represent the coordinates in the reference frame in motion, their respective components of both the magnetic induction and electric field strength shall be directly dependent on the non-primed field components, not on their derivatives. This is achievable when the following hypothesis is assumed

(11)
$$B_1=B_1' \qquad B_2=b_1B_2'+b_2E'_3 \qquad B_3=b_3B_3'+b_4E'_2$$
$$E_1=E_1' \qquad E_2=b_5E'_2+b_6B_3' \qquad E_3=b_7E'_3+b_8B_2'$$

where coefficients b_i generally are functions of variables (x', t'). Following certain heuristic premises, specifically to arrive at the simplest attainable form for the field equations in the primed system and to keep it consistent with Lorentz transformations provided that the system is inertial, we have selected them as follows:

(12)
$$b_1=\frac{1}{D}\left(\frac{1}{c^2}\frac{\partial \varphi_1}{\partial t'}-\frac{\partial \varphi_2}{\partial x'}\right) \qquad b_2=\frac{1}{D}\frac{1}{c^2}\left(\frac{\partial \varphi_1}{\partial t'}-\frac{\partial \varphi_2}{\partial x'}\right)$$
$$b_3=b_1 \qquad b_4=\frac{1}{D}\frac{1}{c^2}\left(\frac{\partial \varphi_2}{\partial t'}-\frac{\partial \varphi_1}{\partial x'}\right) \qquad b_5=b_1$$
$$b_6=b_4 \qquad b_7=b_1 \qquad b_8=-c^2b_4$$
$$D=\frac{1}{c^2}\frac{\partial \varphi_1}{\partial t'}\frac{\partial \varphi_1}{\partial x'}-\frac{\partial \varphi_2}{\partial x'}\frac{\partial \varphi_2}{\partial t'}$$

It allows to separate from equations (9) and (10) the expressions, which correspond to the components of the classic Maxwell's equations; these are grouped on the left sides of the equations (13) and (14):

(13)
$$\frac{\partial B'_3}{\partial y}-\frac{\partial B'_2}{\partial z}-\frac{1}{c^2}\frac{\partial E'_1}{\partial t'}=\frac{1}{c^2}R_2\left(\frac{\partial E'_2}{\partial y}+\frac{\partial E'_3}{\partial z}\right)$$
$$J\frac{\partial B'_1}{\partial z}-\frac{\partial B'_3}{\partial x'}-\frac{1}{c^2}\frac{\partial E'_2}{\partial t'}=\frac{1}{c^2}\frac{\partial}{\partial x'}\left(R_2E'_2\right)-\frac{\partial}{\partial t'}\left(R_1B'_3\right)$$
$$\frac{\partial B'_2}{\partial x'}-J\frac{\partial B'_1}{\partial y}-\frac{1}{c^2}\frac{\partial E'_3}{\partial t'}=\frac{\partial}{\partial t'}\left(R_1B'_2\right)-\frac{1}{c^2}\frac{\partial}{\partial x'}\left(R_2E'_3\right)$$

(14)
$$\frac{\partial E'_3}{\partial y}-\frac{\partial E'_2}{\partial z}+\frac{\partial B'_1}{\partial t'}=-R_2\left(\frac{\partial B'_2}{\partial y}+\frac{\partial B'_3}{\partial z}\right)$$
$$J\frac{\partial E'_1}{\partial z}-\frac{\partial E'_3}{\partial x'}+\frac{\partial B'_2}{\partial t'}=\frac{\partial}{\partial x'}\left(R_2B'_2\right)-\frac{\partial}{\partial t'}\left(R_1E'_3\right)$$
$$\frac{\partial E'_2}{\partial x'}-J\frac{\partial E'_1}{\partial y}+\frac{\partial B'_3}{\partial t'}=\frac{\partial}{\partial x'}\left(R_2B'_3\right)-\frac{\partial}{\partial t'}\left(R_1E'_2\right)$$

where:

(15)
$$R_1=b_2\frac{\partial \varphi_1}{\partial x'}+b_7\frac{\partial \varphi_2}{\partial x'}=\frac{1}{D}\left(\left(\frac{\partial \varphi_1}{c\,\partial x'}\right)^2-\left(\frac{\partial \varphi_2}{\partial x'}\right)^2-\frac{J}{c^2}\right)$$
$$R_2=b_8\frac{\partial \varphi_1}{\partial t'}+b_1\frac{\partial \varphi_1}{\partial t'}=\frac{1}{D}\left(\left(\frac{\partial \varphi_1}{c\,\partial t'}\right)^2-\left(\frac{\partial \varphi_1}{\partial t'}\right)^2+J\right)$$

Electromagnetic field equations in harmonic coordinates

As it can be easily observed, equations (12) and (13) become elegantly simple, provided that functions φ_1, φ_2 satisfy the relations:

(16)
$$\frac{\partial \varphi_2}{\partial t'}=\frac{\partial \varphi_1}{\partial x'}, \quad \frac{\partial \varphi_2}{\partial x'}=\frac{1}{c^2}\frac{\partial \varphi_1}{\partial t'}, \quad \frac{\partial^2 \varphi_i}{\partial x'^2}-\frac{1}{c^2}\frac{\partial^2 \varphi_i}{\partial t'^2}=0$$

which means that they are solutions to the wave equation:

(17)
$$\frac{\partial^2 \varphi_i}{\partial x'^2}-\frac{1}{c^2}\frac{\partial^2 \varphi_i}{\partial t'^2}=0, \qquad i=1,\ 2$$

Coordinates (x', t') for which the required relations are met, are called harmonic coordinates; their role in the theory of gravity by Einstein-Hilbert [1] is also very important. For this transformation class the electromagnetic field components (11) are transformed into the primed coordinates with relations as :

(18)
$$B_1'=B_1 \qquad B_2'=\frac{\partial \varphi_2}{\partial t'}B_2+\frac{\partial \varphi_1}{\partial x'}E_3 \qquad B_3'=\frac{\partial \varphi_2}{\partial t'}B_3-\frac{\partial \varphi_1}{\partial x'}E_2$$

(19)
$$E_1'=E_1 \qquad E_2'=\frac{\partial \varphi_2}{\partial t'}E_2-c^2\frac{\partial \varphi_1}{\partial x'}B_3 \qquad E_3'=\frac{\partial \varphi_2}{\partial t'}E_3+c^2\frac{\partial \varphi_1}{\partial x'}B_2$$

whereas the field equations take a form strikingly similar to the classic Maxwell's equations, namely

(20)
$$J\frac{\partial B_1'}{\partial z'}-\frac{\partial B_3'}{\partial x'}=\frac{1}{c^2}\frac{\partial E_2'}{\partial t'}$$
$$\frac{\partial B_2'}{\partial x'}-J\frac{\partial B_1'}{\partial y'}=\frac{1}{c^2}\frac{\partial E_3'}{\partial t'}$$
$$\frac{\partial B_3'}{\partial y'}-\frac{\partial B_2'}{\partial z'}=\frac{1}{c^2}\frac{\partial E_1'}{\partial t'}$$

(21)
$$J\frac{\partial E_1'}{\partial z}-\frac{\partial E'_3}{\partial x'}=-\frac{\partial B'_2}{\partial t'}$$
$$\frac{\partial E_2'}{\partial x'}-J\frac{\partial E'_1}{\partial y}=-\frac{\partial B'_3}{\partial t'}$$
$$\frac{\partial E'_3}{\partial y'}-\frac{\partial E'_2}{\partial z'}=-\frac{\partial B'_1}{\partial t'}$$

(22)
$$\frac{\partial B_1'}{\partial x'}+\frac{\partial B_2'}{\partial y'}+\frac{\partial B_3'}{\partial z'}=0$$

(23)
$$\frac{\partial E_1'}{\partial x'}+\frac{\partial E_2'}{\partial y'}+\frac{\partial E_3'}{\partial z'}=0$$

Lorentz transformations, for which $J = 1$, provide a particular case for this coordinate transformation class.

Electromagnetic wavefront equation in the primed coordinates

The problem of determining the electromagnetic wave velocity in non-inertial systems is seldom in literature, and individual standpoints authors report are not always clear and unique [2]. The considerations presented in this paper allow to develop the formula for the velocity of the plane electromagnetic wavefront in a non-inertial system being in a linear motion with respect to an arbitrary inertial system in the direction parallel to the direction the wave propagates along.

A general equation describing how the electromagnetic wavefront propagates, which applies also to gravitational waves [1], has the form:

(24) $$\left(\frac{1}{\mathrm{c}}\frac{\partial\omega}{\partial t}\right)^2-\left[\left(\frac{\partial\omega}{\partial x}\right)^2+\left(\frac{\partial\omega}{\partial y}\right)^2+\left(\frac{\partial\omega}{\partial z}\right)^2\right]=0$$

where: $\omega(x, y, z, t) = 0$ - stands for the surface equation of the wavefront.

For the plane wave:

(25) $\omega(x,t)=0$

(26) $$\left(\frac{1}{\mathrm{c}}\frac{\partial\omega}{\partial t}\right)^2-\left(\frac{\partial\omega}{\partial x}\right)^2=0$$

hence:

(27) $$\frac{1}{\mathrm{c}}\frac{\partial\omega}{\partial t}\pm\frac{\partial\omega}{\partial x}=0$$

By applying transformation (7) we obtain

(28) $$\left(\frac{1}{\mathrm{c}}\frac{\partial\varphi_1}{\partial x'}+\frac{\partial\varphi_2}{\partial x'}\right)\frac{\partial\omega}{\partial t'}\pm\left(\frac{1}{\mathrm{c}}\frac{\partial\varphi_1}{\partial t'}+\frac{\partial\varphi_2}{\partial t'}\right)\frac{\partial\omega}{\partial x'}=0$$

which can be also written as

(29) $$\frac{\partial\omega}{\partial t'}=\pm\frac{\frac{\partial}{\partial t'}\left(\varphi_1+\mathrm{c}\varphi_2\right)}{\frac{\partial}{\partial x'}\left(\varphi_1+\mathrm{c}\varphi_2\right)}\frac{\partial\omega}{\partial x'}$$

On the surface of the wave described in the primed system by the relation $\omega(x',t')=0$ the variables x' and t' are interdependent, whereas

(30) $$v=\frac{dx'}{dt'}$$

denotes the velocity of the wavefront. To determine its value we applied the characteristics method [4]. The solution to the equation (28) can be expressed with $\omega=\omega(\eta)$, where $\eta = \eta(x',t')$ describes characteristics, that is for each characteristics $\eta = \mathrm{const}$. Thus:

(31) $$\frac{d\eta}{dt'}=\frac{\partial\eta}{\partial t'}+\frac{\partial\eta}{\partial x'}\frac{dx'}{dt'}=0$$

Noting that characteristics must satisfy an equation analogous to (29), as

(32) $$\frac{\partial\omega}{\partial t'}-R\frac{\partial\omega}{\partial x'}=0 \quad\Leftrightarrow\quad \frac{\partial\omega}{\partial\eta}\left(\frac{\partial\eta}{\partial t'}+v\frac{\partial\eta}{\partial x'}\right)=0$$

we find the following formula for the wavefront velocity

(33) $$v=\pm\frac{\frac{\partial}{\partial t'}\left(\varphi_1+\mathrm{c}\varphi_2\right)}{\frac{\partial}{\partial x'}\left(\varphi_1+\mathrm{c}\varphi_2\right)}$$

It is worth observing that for harmonic variables $v=\pm\mathrm{c}$.

Similarly, such considerations can be performed for a spherical wave, as in a spherical coordinate system the equation (26) will be formulated identically, with x coordinate replaced by r.

Conclusions

a) Having transformed the inertial frame time-space coordinates $(x,\ t)$ under one-to-one arbitrary transformation into $(x',\ t')$ the electromagnetic field equations can be formulated with the relations (13) – (14).
b) The electromagnetic field equations (see (16)) in harmonic coordinate system take an especially simple and elegant form, nearly identical to the one in the inertial reference frame (see (20) – (23)).
c) In an arbitrary reference frame remaining in arbitrary linear motion with respect to the inertial reference frame the velocity of a plane electromagnetic wave propagating in parallel to the relative velocity of the reference frames can be given by (33). Shall the coordinates of the reference frame in motion be harmonic ones, such velocity equals the velocity of light in a vacuum in the inertial reference frame.

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