Stanisław APANASEWICZ[1], Stanisław PAWŁOWSKI[1], Jolanta PLEWAKO[2]

Politechnika Rzeszowska, Katedra Elektrodynamiki i Układów Elektromaszynowych (1), Katedra Energoelektroniki i Elektroenergetyki (2)

Transformations of space-time coordinates and electromagnetic field equations in non-inertial reference frames

***Abstract**. In the presented paper a procedure allowing to identify a transformation for space-time coordinates from inertial to non-inertial reference frame in a linear motion (a generalised Lorentz's transformation), as well as the electromagnetic field forms in this reference frame.*

Streszczenie.** W prezentowanej pracy zaproponowano procedurę znajdowania transformacji współrzędnych czasoprzestrzennych z inercjalnego do nieinercjalnego układu odniesienia poruszającego się ruchem prostoliniowym (uogólnienie przekształceń Lorentza) oraz postaci równań pola elektromagnetycznego w tym układzie. **Znajdowania transformacji współrzędnych czasoprzestrzennych z inercjalnego do nieinercjalnego układu odniesienia poruszającego się ruchem prostoliniowym

Keywords. non-inertial reference frames, equations of electrodynamics, Lorentz transformations, Liénard-Wiechert formulas
Słowa kluczowe. układy nieinercjalne, równania elektrodynamiki, transformacje Lorentza, wzory Liénarda-Wiecherta

Introduction

In the presented paper a procedure allowing to identify a transformation for spacetime coordinates from inertial to non-inertial reference frame in a linear motion (a generalised Lorentz's transformation), as well as the electromagnetic field forms in this reference frame. The method is based on the analysis of the electromagnetic field generated by a charge travelling at variable speed, defined with the formulas for retarded Liénard – Wiechert potentials. The paper continues research reported elsewhere [1], [2].

Procedure for finding a transformation for spacetime coordinates

As a starting point for further considerations we adopted the analysis of the problem on calculating the distribution of an electromagnetic field emitted by a point charge Q travelling through a vacuum in a linear motion at usually variable velocity $\boldsymbol{u}$ with respect to an arbitrary inertial reference frame, called hereafter *the resting system*. Within this system we define Cartesian spacetime coordinates adopting OX axis to be parallel to the charge velocity vector. At first we need to specify transformation relationships transposing these coordinates to the coordinate system defined for Q charge, hereafter called *the system in motion*, whose axes are parallel to the axes of the resting system (each respectively), whereas its origin is placed at Q charge.

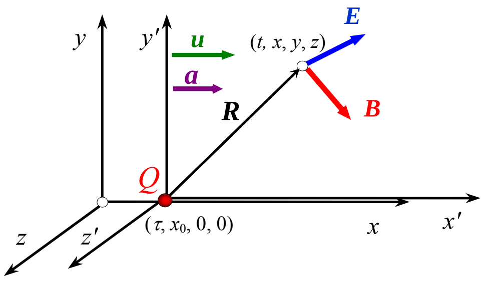

Fig. 1. The systems under analysis

Having assumed that $y'=y$, $z'=z$, we seek the functions

$$x=\varphi_1(x',t'),\quad t=\varphi_2(x',t')$$

$$x'=\xi_1(x,t),\quad t'=\xi_2(x,t)$$

In this specified physical case the vectors for position $\boldsymbol{r}_0$, velocity $\boldsymbol{u}$ and acceleration $\boldsymbol{a}$ of the Q charge in the resting system at τ time can be written as:

$$\boldsymbol{r}_0=[x_0(\tau),0,0],\quad \boldsymbol{u}=[u(\tau),0,0]\quad \boldsymbol{a}=[a(\tau),0,0]$$

where:

$$u=\frac{\partial x_0}{\partial \tau},\quad a=\frac{\partial u}{\partial \tau}$$

With no limitations to problem generality we have assumed that at τ time the origins of both systems overlap.

It could be proven with Lienard-Wiechert formulas that x component of the electric field E_1 on the OX ($y=z=0$) axis emitted by Q charge at t time can be expressed by the relation [1]:

$$E_1(x,t)=\frac{Q}{4\pi\varepsilon_0}\frac{1}{[(x-x_0(\tau))g(\tau)]^2}$$

where:

$$g(\tau)=\sqrt{\frac{\mathrm{c}-u(\tau)}{\mathrm{c}+u(\tau)}}$$

As the electromagnetic signals propagate within the resting system with the velocity c, thus the t time (5) is retarded with respect to τ time by a time fraction R/c, hence:

$$\tau=t-\frac{x-x_0(\tau)}{\mathrm{c}}$$

It is widely known that for transformations between two inertial reference frames the field components that are parallel to the direction of the relative velocity of the systems are invariable [3], [4]. Guided by this premise we adopt a postulate that in the considered case we also have:

$$E_1'(x',t')=E_1(x,t)$$

which along with (5) and Coulomb-like dependence of this component on the distance in the resting system means that

$$x'=(x-x_0(\tau))g(\tau)$$

By analogy we can write the inverse relations to (7) and (9):

$$\tau'=t'-\frac{x'+x_0'(\tau')}{\mathrm{c}}$$

$$x=(x'+x_0'(\tau'))g'(\tau')$$

where $x_0'(\tau')$ represents the distance between the origins of the coordinate systems measured in the system in motion at τ' time; similarly as for τ time we assume that at $\tau'=0$ the charge Q remains at the origin of the resting system. By the axiom on relative velocities between two arbitrary reference frames providing their equal value and opposite direction, that is:

$$u'(\tau') \equiv -\frac{\partial x_0'}{\partial \tau'} = -u(\tau)$$

we obtain:

$$g' = \frac{1}{g} = \sqrt{\frac{\mathrm{c}+u(\tau)}{\mathrm{c}-u(\tau)}}$$

From (9) and (11) we have:

$$x_0'(\tau') = x_0(\tau)g(\tau)$$

Having differentiated this equity by τ and following few simple operations we provide a relation:

$$\tau' = F(\tau) = \int_0^{\tau} \left(g(\tau) - \frac{x_0(\tau)}{u(\tau)} \frac{dg}{d\tau} \right) d\tau$$

By applying the equations (7), (9), (10), (11), (14), (15) the sought transformation relations between the coordinates of the two systems (1), (2) can be finally specified. The procedure runs as follows:

1. Defining the relation $\tau = \tau\ (t - x/\mathrm{c})$ by solving the algebraic equation (7) with the function $x_0 = x_0(\tau)$ given.

2. Substituting $\tau = \tau\ (t - x/\mathrm{c})$ to (9) provides the first of the relations sought for $x' = \xi_1(x,t)$.

3. Defining the relation $\tau' = F(\tau)$ on the grounds of (15) and the inverse relation $\tau = F'(\tau')$.

4. Defining the relation $x_0' = x_0'(\tau')$ by substituting it to (14) $\tau = F'(\tau')$.

5. Defining the relation $\tau' = \tau'(t' - x'/\mathrm{c})$ by solving the equation (10).

6. By substituting $\tau' = \tau'(t' - x'/\mathrm{c})$ to (11) we find the second relation sought $x = \varphi_1(x',t')$.

7. Solving the set of equations $x = \varphi_1(x',t')$, $x' = \xi_1(x,t)$ with respect to t and t' completes the transformation set (1), (2) with functions φ_2, ξ_2.

The main difficulty of the procedure lies in the necessity of solving non-linear algebraic equations (7) and (10), usually quite complex in form, i.e. dependent on the function $x_0 = x_0(\tau)$ which describes how the system in motion travels with respect to the resting one, therefore it is most often performed numerically. In practice it may prove helpful for calculations that within relatively wide range of velocities, i.e. up to ca 0.8 c, the g function expressed with formula (6) can be safely approximated with a linear function (Fig. 2).

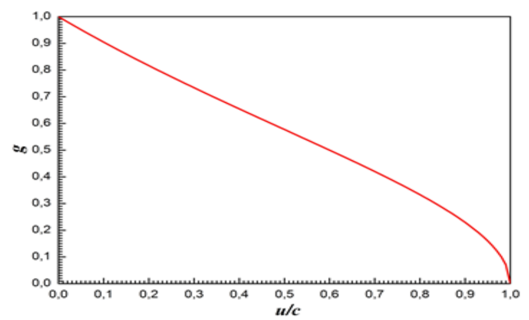

Fig. 2. Plot of the function $g = g(u/\mathrm{c})$ expressed with (6)

Differential relations

To obtain the electromagnetic field equations for the system in motion it is necessary to start with defining differential operators with respect to x, t coordinates as variables of x', t' coordinates. On the ground of (1), (2) we can generally write:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \cdot \frac{\partial \xi_1}{\partial x} + \frac{\partial}{\partial t'} \cdot \frac{\partial \xi_2}{\partial x}$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x'} \cdot \frac{\partial \xi_1}{\partial t} + \frac{\partial}{\partial t'} \cdot \frac{\partial \xi_2}{\partial t}$$

By using (9) we arrive at:

$$\frac{\partial \xi_1}{\partial x} = \left(1 - u\frac{\partial \tau}{\partial x}\right) + (x - x_0)\frac{dg}{d\tau}\frac{\partial \tau}{\partial x}$$

$$\frac{\partial \xi_1}{\partial t} = -ug\frac{\partial \tau}{\partial x} + (x - x_0)\frac{dg}{d\tau}\frac{\partial \tau}{\partial t}$$

while (6) and (11) used give:

$$\frac{dg}{d\tau} = -\frac{\gamma\, a}{\mathrm{c}\left(1 + \frac{u}{\mathrm{c}}\right)}, \qquad \gamma = \frac{1}{\sqrt{1 - \frac{u^2}{\mathrm{c}^2}}}$$

$$\frac{\partial \tau}{\partial x} = -\frac{1}{\mathrm{c}\left(1 - \frac{u}{\mathrm{c}}\right)}, \qquad \frac{\partial \tau}{\partial t} = \frac{1}{1 - \frac{u}{\mathrm{c}}}$$

Having substituted (18, 19) into (17) we get :

$$\frac{\partial \xi_1}{\partial x} = \gamma + \frac{1}{\mathrm{c}}L_1, \qquad \frac{\partial \xi_1}{\partial t} = -\gamma\, u - L_1$$

where:

$$L_1 = \frac{(x - x_0)\gamma^3 a}{\mathrm{c}}$$

Making use of (10) we calculate derivation if the function ξ_2, arriving at:

$$\frac{\partial \xi_2}{\partial t} = \frac{1}{\mathrm{c}}\frac{\partial \xi_1}{\partial t} + \left(1 + \frac{u}{\mathrm{c}}\right)\frac{\partial \tau'}{\partial t}$$

and with the formula (14):

$$\frac{\partial x_0'}{\partial t} = \frac{\partial x_0'}{\partial \tau} \cdot \frac{\partial \tau}{\partial t} = \frac{\partial (x_0 g)}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{1}{1 - \frac{u}{\mathrm{c}}}\left[ug - \frac{x_0 \gamma\, a}{\mathrm{c}\left(1 + \frac{u}{\mathrm{c}}\right)}\right]$$

and

$$\frac{\partial x_0'}{\partial t} = \frac{\partial x_0'}{\partial \tau'} \cdot \frac{\partial \tau'}{\partial t} = -u\frac{\partial \tau'}{\partial t}$$

From this equation we can calculate $\frac{\partial \tau'}{\partial t}$ and then substitute it to (22) which yields:

$$\frac{\partial \xi_2}{\partial t} = \frac{1}{\mathrm{c}}\frac{\partial \xi_1}{\partial t} + \left(1 + \frac{u}{\mathrm{c}}\right)\left[\gamma - \frac{x_0 \gamma^3\, a}{\mathrm{c}u}\right] = \gamma - L_2$$

$$L_2 = \frac{xu + x_0 \mathrm{c}}{\mathrm{c}^2 u}\gamma^3\, a$$

Similarly, we obtain:

$$\frac{\partial \xi_2}{\partial x} = -\frac{u\gamma}{\mathrm{c}^2} + \frac{1}{\mathrm{c}}L_2$$

And finally:

$$\frac{\partial}{\partial x} = \left(\gamma + \frac{1}{\mathrm{c}}L_1\right)\frac{\partial}{\partial x'} + \left(-\gamma + \frac{1}{\mathrm{c}}L_2\right)\frac{\partial}{\partial t'}$$

$$\frac{\partial}{\partial t} = -(\gamma u + L_1)\frac{\partial}{\partial x'} + (\gamma - L_2)\frac{\partial}{\partial t'}$$

Transformations for function class more general than the ones resulting from formulas (4), (5) are presented in [2].

Electromagnetic field equations in the system in motion

By applying the operators (27), (28) to sourceless Maxwell's equations:

$$\operatorname{rot} \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \quad \operatorname{rot} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

their explicit form in the system in motion can be finally written as:

$$\frac{\partial B'_3}{\partial y} - \frac{\partial B'_2}{\partial z} - \frac{1}{\mathrm{c}^2}\frac{\partial E'_1}{\partial t'} = \frac{g}{\mathrm{c}^2}\left(L_1 \frac{\partial E'_1}{\partial x'} + L_2 \frac{\partial E'_1}{\partial t'} \right)$$

$$\frac{\partial B'_1}{\partial z} - \frac{\partial B'_3}{\partial x'} - \frac{1}{\mathrm{c}^2}\frac{\partial E'_2}{\partial t'} = \frac{g}{\mathrm{c}}\left(L_1 \frac{\partial F_1}{\partial x'} + L_2 \frac{\partial F_1}{\partial t'} \right) + M \cdot F_1$$

$$\frac{\partial B'_2}{\partial x'} - \frac{\partial B'_1}{\partial y} - \frac{1}{\mathrm{c}^2}\frac{\partial E'_3}{\partial t'} = -\frac{g}{\mathrm{c}}\left(L_1 \frac{\partial F_2}{\partial x'} + L_2 \frac{\partial F_2}{\partial t'} \right) - M \cdot F_2$$

$$\frac{\partial E'_3}{\partial y} - \frac{\partial E'_2}{\partial z} + \frac{\partial B'_1}{\partial t'} = g\left(L_1 \frac{\partial B'_1}{\partial x'} + L_2 \frac{\partial \partial}{\partial t'} \right)$$

$$\frac{\partial E'_1}{\partial z} - \frac{\partial E'_3}{\partial x'} + \frac{\partial B'_2}{\partial t'} = g\left(L_1 \frac{\partial F_2}{\partial x'} + L_2 \frac{\partial F_2}{\partial t'} \right) + \frac{1}{\mathrm{c}} M \cdot F_2$$

$$\frac{\partial E'_2}{\partial x'} - \frac{\partial E'_1}{\partial y} + \frac{\partial B'_3}{\partial t'} = g\left(L_1 \frac{\partial F_1}{\partial x'} + L_2 \frac{\partial F_1}{\partial t'} \right) + \frac{1}{\mathrm{c}} M \cdot F_1$$

where:

$$B'_1 = B_1, \quad B'_2 = \gamma\left(B_2 + \frac{u}{c^2} E_3 \right), \quad B'_3 = \gamma\left(B_3 + \frac{u}{c^2} E_2 \right)$$

$$E'_1 = E_1, \quad E'_2 = \gamma\left(E_2 - uB_3 \right), \quad E'_3 = \gamma\left(E_3 + uB_2 \right)$$

that is we assume field components not to be dependent on acceleration and to transform according to standard Lorentz transformation [3], and:

$$F_1 = B'_3 - \frac{1}{\mathrm{c}} E'_2, \qquad F_2 = B'_2 + \frac{1}{\mathrm{c}} E'_3$$

$$M = \frac{\gamma^2 a\left(\mathrm{c}u - \gamma^3 x_0 a\right)}{\mathrm{c}^2\left(\mathrm{c}u - \gamma^2 x_0 a\right)}$$

To derive equations (30), (31) the following relations have been used

$$\frac{\partial \gamma}{\partial x'} = -\frac{1}{\mathrm{c}}\frac{\partial \gamma}{\partial t'} = -\frac{\gamma^4 ua}{\mathrm{c}^3 K}$$

$$\frac{\partial(\gamma u)}{\partial x'} = -\frac{1}{\mathrm{c}}\frac{\partial(\gamma u)}{\partial t'} = -\frac{\gamma^4 a}{\mathrm{c} K}$$

where $K = 1 - \dfrac{\gamma^2 x_0\, a}{\mathrm{c}u}$

Summary

The presented procedure for finding spacetime coordinates transformation between an inertial system and a system travelling in a linear motion with respect to the inertial one, proved to be rather complex. The main difficulty lies in obligatory solving non-linear algebraic equations generally intricate in their form, which customarily requires numerical methods to be employed.

The derived formulas (27, 28) expressing differentiation operation with respect to the inertial system coordinates as dependent on the differentiation operation in the non-inertial system, facilitated finding the equations that describe the electromagnetic field in the system in motion.

Under the assumed relation between the components of magnetic and electric field vectors in both systems to be taking a form analogous to Lorentz transformation, the electromagnetic field equations that held in the non-inertial system (30), (31) were found. In the equations parts dependent on acceleration, expressions that are dependent on some recurrent combinations of transverse field components draw attention (34); their physical interpretation requires further research and analyses.

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***Authors**: prof. dr Stanisław Apanasewicz, Department of Electrodynamics and Electrical Machine Systems, W. Pola 2, 35-959 Rzeszów, E-mail: sapanas@prz.edu.pl, dr hab. Stanisław Pawłowski, Department of Electrodynamics and Electrical Machine Systems, W. Pola 2, 35-959 Rzeszów, E-mail: spawlo@prz.edu.pl, dr inż. Jolanta Plewako, Department of Power Electronics and Power Engineering, W. Pola 2, 35-959 Rzeszów, E-mail: jplewako@prz.edu.pl.*