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Generalizing Dempster's combination rule to fuzzy sets

***Abstract.** The fuzzy and imprecise information always exist in real systems. Several attempts have been made to generalize the Dempster-Shafer (D-S) evidence theory to deal with fuzzy sets. In order to combine bodies of evidence that may contain vague information, Dempster's combination rule was extended to fuzzy sets in the evidential reasoning. In this paper, a new definition of the weight between two fuzzy sets is described, and the improved extension combination rule of the evidence theory on fuzzy sets is put forward. Compared with other generalization of Dempster's combination rules, the results of the numerical experiments show that the new combination rule in this paper can acquire more changing information to the change of fuzzy focal elements more effectively.*

Streszczenie** W rzeczywistych systemach zwykle występuje informacja rozmyta i nieprecyzyjna. Istnieje szereg prób uogólnienia teorii Dempster'a-Shafer'a w zastosowaniu do zbiorów rozmytych. W wywodzie dowodowym, w celu dołączenia fragmentu danych, które mogą zawierać informację nieprecyzyjną, rozciągnięto regułę kombinacji Dempster'a na zbiory rozmyte. W opracowaniu opisano nową definicję wagi między dwoma zbiorami rozmytymi oraz przedstawiono udoskonalone rozszerzenie kombinacyjnej reguły badanej teorii na zbiory rozmyte. W porównaniu z innymi uogólnieniami reguł kombinacji Dempster'a, wyniki eksperymentalne pokazują, że nowa, przedstawiona w opracowaniu, reguła kombinacji może bardziej skutecznie uzyskać więcej zmieniających się informacji przy zmianie centralnych elementów zbioru. **Uogólnienie reguły kombinacji Dempster'a do zbiorów rozmytych

Keywords: Information fusion, Dempster's combination rule, Fuzzy combination rule, Fuzzy sets
Słowa kluczowe: Połączenie informacji, Reguła kombinacji Dempster'a, Reguła kombinacji rozmytych, Zbiory rozmyte

1. Introduction

One of trends of the contemporary scientific and technological development is that it is required to perform quantification and mathematization in each disciplinary field, and it is also required to quantify and mathematize the fuzzy concept or phenomenon urgently, which impels people to search for a mathematic method of processing fuzzy data. As capably of disposing the uncertainty induced by ignorance, D-S evidence theory[1-3] adopts belief function not the probability as the measurement, restraining the probability of some incidents to establish belief function without specifying the probability which is hard to obtain. However, the relationship among the sets of the intra-domain and those of on-domain of the evidence theory is the relationship of "belonging to or not belonging to". It is hard to dispose the phenomenon of "both this and that" of fuzziness. The disposal of fuzzy sets has obvious advantages over uncertainty of fuzzy. Therefore, D-S evidence theory could be generalized to fuzzy sets and the information about inaccuracy and fuzziness could be represented and disposed by using the advantages of D-S evidence theory and fuzzy sets. This is extraordinarily significant to expand the application range of evidence theory.

To combine bodies of evidence that may contain vague information, Dempster's combination rule was extended to fuzzy sets. Ishizuka et al.[4] extended Dempster's combination rule to fuzzy based on the degree of fuzzy intersection of two fuzzy sets. Yen[5] used cross product operation and normalization process to extend Dempster's combination rule to fuzzy sets. Yang et al.[6] extended Dempster's combination rule to fuzzy sets by taking into account the weight of two fuzzy sets. But those fuzzy combination rules cannot catch the actual focal element change information effectively and are insensitive to the subtle changes. In this paper, a new generalization of Dempster's combination rule to fuzzy sets is proposed. In Section 2, we briefly review some existing extensions of Dempster's combination rule. In Section 3, we propose a new method of extending Dempster's combination rule to fuzzy sets based on the weight between two fuzzy sets. In Section 4, we make the comparisons with other existing extensions. Conclusions are then given in Section 5.

2. Different generalizations of Dempster's combination rule

Let Bel_1 and Bel_2 be two belief functions of crisp sets over the same frame of discernment. If m_1 and m_2 are the basic probability assignments (BPAs) of Bel_1 and Bel_2 respectively, then the combined BPA for a crisp set C is computed by Dempster's combination rule as:

(1) $$m_1 \oplus m_2(C) = \frac{\sum_{A\cap B=C} m_1(A)m_2(B)}{1-\sum_{A\cap B=\varnothing} m_1(A)m_2(B)}$$

Based on the degree $J(\tilde{A},\tilde{B})$ of fuzzy intersection of two fuzzy sets, Ishizuka et al.[4] extended Dempster's combination rule to fuzzy sets as:

(2) $$m_1 \oplus m_2(\tilde{C}) = \frac{\sum_{\tilde{A}\cap\tilde{B}=\tilde{C}} J(\tilde{A},\tilde{B})m_1(\tilde{A})m_2(\tilde{B})}{1-\sum_{\tilde{A},\tilde{B}}(1-J(\tilde{A},\tilde{B}))m_1(\tilde{A})m_2(\tilde{B})}$$

where $J(\tilde{A},\tilde{B}) = \dfrac{\max_x \mu_{\tilde{A}\cap\tilde{B}}(x)}{\min\{\max_x \mu_{\tilde{A}}(x), \max_x \mu_{\tilde{B}}(x)\}}$.

Yen[5] used cross product operation and normalization process to extend Dempster's combination rule to fuzzy sets as follows:

1) Cross-product

(3) $$m'(\tilde{C}) = m_1 \otimes m_2(\tilde{C}) = \sum_{\tilde{A}\cap\tilde{B}=\tilde{C}} m_1(\tilde{A})m_2(\tilde{B})$$

2) Normalization

(4) $$N[m'](\tilde{D}) = \frac{\sum_{\tilde{C}=\tilde{D}} \max_x \mu_{\tilde{C}}(x)m'(\tilde{C})}{1-\sum_{\tilde{C}}(1-\max_x \mu_{\tilde{C}}(x))m'(\tilde{C})}$$

Yang et al.[6] extended Dempster's combination rule to fuzzy sets by taking into account the weight $W(\tilde{C},\tilde{A})$ of two fuzzy sets as:

(5) $$m_1 \oplus m_2(\tilde{C}) = \frac{\sum_{\tilde{A}\cap\tilde{B}=\tilde{C}} W(\tilde{C},\tilde{A})m_1(\tilde{A})W(\tilde{C},\tilde{B})m_2(\tilde{B})}{1-\sum_{\tilde{A},\tilde{B}}(1-W(\tilde{A}\cap\tilde{B},\tilde{A})W(\tilde{A}\cap\tilde{B},\tilde{B}))m_1(\tilde{A})m_2(\tilde{B})}$$

where $W(\tilde{C},\tilde{A}) = \dfrac{|\tilde{C}|}{|\tilde{A}|}$, $|\tilde{C}| = \sum_x \mu_{\tilde{C}}(x)$, $|\tilde{A}| = \sum_x \mu_{\tilde{A}}(x)$.

Ishizuka et al.[4] adopted the max and min operators in the definition of $J(\tilde{A},\tilde{B})$. The degree is decided by some critical points. Thus the combination results not sensitive to the changes in focal element information. Yen's method[5] cannot gain the change information of the fuzzy focal element effectively for directly extending Dempster's combination rule and adopting the final result of the normalization process. In order to dispose uncertain and fuzzy information effectively, we propose another extension of Dempster's combination rule to fuzzy sets by constructing a weight variable $\omega(\tilde{C},\tilde{A})$, which expresses the weight of contribution to the fuzzy set $\tilde{C}$ from a focal element $\tilde{A}$.

3. A new generalizing Dempster's combination rule to fuzzy sets

Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets in X ={ x_1 , x_2 , $\cdots$, x_n }. Let $\tilde{A}$ =($\mu_{\tilde{A}}(x_1)$ / x_1 , $\mu_{\tilde{A}}(x_2)$ / x_2 , $\cdots$, $\mu_{\tilde{A}}(x_n)$ / x_n), $\tilde{B}$ =($\mu_{\tilde{B}}(x_1)$ / x_1 , $\mu_{\tilde{B}}(x_2)$ / x_2 , $\cdots$, $\mu_{\tilde{B}}(x_n)$ / x_n). Functions $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$ are called the membership functions of the fuzzy sets $\tilde{A}$ and $\tilde{B}$ respectively. The definition of weight $\omega(\tilde{C},\tilde{A})$ is as below:

(6)
$$\begin{cases} \omega(\tilde{C},\tilde{A}) = \dfrac{1-\dfrac{1}{|\Theta|}\sum_i |\mu_{\tilde{C}}(x_i)-\mu_{\tilde{A}}(x_i)|}{1+\dfrac{1}{|\Theta|}\sum_i |\mu_{\tilde{C}}(x_i)-\mu_{\tilde{A}}(x_i)|}, & C \neq \varnothing \\ \omega(\tilde{C},\tilde{A}) = 0, & C = \varnothing \end{cases}$$

where $|\Theta|$ is the number of elements of Θ. For a fuzzy set $\tilde{C}$, the combined fuzzy BPA $m_1 \oplus m_2$ of two fuzzy BPAs m_1 and m_2 is defined as:

(7) $m(\tilde{C}) = m_1 \oplus m_2(\tilde{C})$ =
$$\begin{cases} \dfrac{\sum_{\tilde{A}_i \cap \tilde{B}_j = \tilde{C}} \omega(\tilde{C},\tilde{A}_i) m_1(\tilde{A}_i) \omega(\tilde{C},\tilde{B}_j) m_2(\tilde{B}_j)}{1-\sum_{\tilde{A}_i,\tilde{B}_j}(1-\omega(\tilde{A}_i\cap\tilde{B}_j,\tilde{A}_i)\omega(\tilde{A}_i\cap\tilde{B}_j,\tilde{B}_j)) m_1(\tilde{A}_i) m_2(\tilde{B}_j)}, & \tilde{C} \neq \varnothing \\ 0, & \tilde{C} = \varnothing \end{cases}$$

Our fuzzy combination rule is the same as the methods proposed by Yen and Yang et al., not satisfying the associative. To be capable of performing many evidence combinations, the strategy proposed by Yang[6] is adopted in this paper to realize the multiple evidence combination problems by steps: firstly, the Dempster's combination rule is used to perform combination consecutively to gain the new BPAs of the fuzzy focal element; secondly, the weights between two fuzzy sets are adopted to perform normalization to make the summation of fuzzy BPAs is equal to one. The two steps in the combination rule are formulated as follows:

Step 1:

(8) $$m_1 \oplus \ldots \oplus m_n(\tilde{C}) = \sum_{\tilde{A}_1 \cap \ldots \cap \tilde{A}_n = \tilde{C}} m_1(\tilde{A}_1) \ldots m_n(\tilde{A}_n) = m_{1\ldots n}(\tilde{C})$$

Step 2:

(9) $N[m_1 \oplus \ldots \oplus m_n](\tilde{B})$ =
$$\frac{\sum_{\tilde{C}=\tilde{B}} \omega(\tilde{B},\tilde{A}_1)\ldots\omega(\tilde{B},\tilde{A}_n) m_{1\ldots n}(\tilde{C})}{1-\sum_{\tilde{A}_1,\ldots,\tilde{A}_n}(1-\omega(\tilde{B},\tilde{A}_1)\ldots\omega(\tilde{B},\tilde{A}_n)) m_{1\ldots n}(\tilde{C})}$$

4. Numerical examples

Let Θ ={2,3,4,5,6} and let all focal elements be as follows:

$\tilde{A}_1$ ={0.75/2, 0.5/3, 0.75/4, 1/5}

$\tilde{A}_2$ ={0.5/3, 1/4, 0.5/5}

$\tilde{A}_3$ ={0.25/2, 1/3, 0.75/4}

$\tilde{A}_4$ ={0.5/5, 1/6}

$\tilde{A}_5$ ={0.25/2, 1/4, 0.75/6}

Assume that there are two experts assigning the two BPAs m_1 and m_2 listed in Table 1.

Table 1. BPAs m_1 and m_2 of two experts for $\tilde{A}_1 \sim \tilde{A}_5$

Assignments of expert	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_4$	$\tilde{A}_5$
Expert 1 (m_1)	0.6	0.2	0.2	0.0	0.0
Expert 2 (m_2)	0.0	0.4	0.3	0.1	0.2

The four methods are used to combine fuzzy BPAs $m_1 \oplus m_2$. Table 2 lists the combination results.

Table 2. Combined focal elements and BPAs for different methods

Combined focal element	Ishizuka[4]	Yen[5]	Yang[6]	Ours
$\tilde{C}_1 = \tilde{A}_1 \cap \tilde{A}_2$	0.2416	0.2416	0.2851	0.2523
$\tilde{C}_2 = \tilde{A}_1 \cap \tilde{A}_3$	0.1812	0.1812	0.1571	0.1536
$\tilde{C}_3^1 = \tilde{A}_1 \cap \tilde{A}_4$	0.0403	0.0403	0.0078	0.0258
$\tilde{C}_4^1 = \tilde{A}_1 \cap \tilde{A}_5$	0.1208	0.1208	0.0465	0.0664
$\tilde{C}_5 = \tilde{A}_2 \cap \tilde{A}_2$	0.1074	0.1074	0.1862	0.1549
$\tilde{C}_6^1 = \tilde{A}_2 \cap \tilde{A}_3$	0.0604	0.0604	0.0545	0.0635
$\tilde{C}_3^2 = \tilde{A}_2 \cap \tilde{A}_4$	0.0134	0.0134	0.0039	0.0139
$\tilde{C}_7 = \tilde{A}_2 \cap \tilde{A}_5$	0.0537	0.0537	0.0233	0.0344
$\tilde{C}_6^2 = \tilde{A}_3 \cap \tilde{A}_2$	0.0805	0.0805	0.0727	0.0846
$\tilde{C}_8 = \tilde{A}_3 \cap \tilde{A}_3$	0.0604	0.0604	0.1396	0.1162
$\tilde{C}_4^2 = \tilde{A}_3 \cap \tilde{A}_5$	0.0403	0.0403	0.0233	0.0344

$\tilde{C}_1$ ={0.5/3, 0.75/4, 0.5/5}, $\tilde{C}_2$ ={0.25/2, 0.5/3, 0.75/4}, $\tilde{C}_3^1$ = $\tilde{C}_3^2$ ={0.5/5}, $\tilde{C}_4^1$ = $\tilde{C}_4^2$ ={0.25/2, 0.75/4}, $\tilde{C}_5$ ={0.5/3, 1/4, 0.5/5}, $\tilde{C}_6^1$ = $\tilde{C}_6^2$ ={0.5/3, 0.75/4}, $\tilde{C}_7$ ={1/4}, and $\tilde{C}_8$ ={0.25/2, 1/3, 0.75/4}.

According to the results of Table 2, it can be seen that, our combination method can catch the contribution to the newly combined evidence. For instance, as for $\tilde{C}_3^1$ = $\tilde{C}_3^2$, the combined BPA values are not the same, for $\tilde{C}_3^1$ is combined by $m_1(\tilde{A}_1)$ and $m_2(\tilde{A}_4)$ while $\tilde{C}_3^2$ is combined by $m_1(\tilde{A}_2)$ and $m_2(\tilde{A}_4)$.

To compare the validity of different methods in catching information during the process of the evidence combination, we change $\tilde{A}_1$ to $\tilde{A}_1'$, $\tilde{A}_1''$, and $\tilde{A}_1'''$ with

$\tilde{A}_1'$ ={1/2, 0.5/3, 0.75/4, 1/5}

$\tilde{A}_1''$ ={0.5/2, 0.5/3, 0.75/4, 1/5}

$\tilde{A}_1'''$ ={0.75/2, 0.5/3, 0.75/4, 0.5/5}

The changes from $\tilde{A}_1$ to $\tilde{A}_1'$, $\tilde{A}_1''$, and $\tilde{A}_1'''$ only changes the partial membership function values without changing the fuzzy focal element. Therefore, the fuzzy focal elements of the newly combined evidence are still $\tilde{C}_1 \sim \tilde{C}_8$. Based on the same combination method, the change of $\tilde{A}_1$ will lead changes to the BPAs of the combined focal element. How the combined BPA calculated by different methods changes is listed in Table 3.

Table 3. Changes to combined BPA caused by changes in fuzzy focal element

Combined focal element	$\tilde{A}_1 \to \tilde{A}_1'$				$\tilde{A}_1 \to \tilde{A}_1''$				$\tilde{A}_1 \to \tilde{A}_1'''$			
	Ishizuka[4]	Yen[5]	Yang[6]	Ours	Ishizuka[4]	Yen[5]	Yang[6]	Ours	Ishizuka[4]	Yen[5]	Yang[6]	Ours
$\tilde{C}_1$	U	U	D	D	U	U	I	I	I	I	I	I
$\tilde{C}_2$	U	U	D	D	U	U	I	I	I	I	I	I
$\tilde{C}_3^1$	U	U	D	D	U	U	I	I	I	I	I	I
$\tilde{C}_4^1$	U	U	D	D	U	U	I	I	I	I	I	I
$\tilde{C}_5$	U	U	I	I	U	U	D	D	D	D	D	D
$\tilde{C}_6^1$	U	U	I	I	U	U	D	D	D	D	D	D
$\tilde{C}_3^2$	U	U	I	I	U	U	D	D	D	D	D	D
$\tilde{C}_7$	U	U	I	I	U	U	D	D	D	D	D	D
$\tilde{C}_6^2$	U	U	I	I	U	U	D	D	D	D	D	D
$\tilde{C}_8$	U	U	I	I	U	U	D	D	D	D	D	D
$\tilde{C}_4^2$	U	U	I	I	U	U	D	D	D	D	D	D

Note: U, I and D denote unchanged, increased, and decreased, respectively.

According to the results of Table 3, it can be seen that our fuzzy combination rule can catch the change information of the fuzzy focal element further effectively than that of Ishizuka et al. and Yen, thus the combination results are more beneficial to the decision. As for the method proposed by Ishizuka, based on the intersection degree $J(\tilde{A},\tilde{B})$ of the two fuzzy sets, can lead changes to the final combination result only in case of changes to the intersection degree. Yen's method cannot gain the change information of the fuzzy focal element effectively for directly extending Dempster's combination rule and adopting the final result of the normalization process. The fundamental principle of our extension is the same with the proposed by Yang et al. in establishing fuzzy combination rule, only different in the definition of the weight. Yang's weight relies on the membership degree sum of the fuzzy sets. Our weight is based on the sum of the difference in the membership degree of two fuzzy sets, more capable of catching the difference between the two fuzzy sets than the Yang's method (see Table 2).

5. Conclusions

In this paper, the research on extension of Dempster's combination rule to fuzzy sets was considered. A new definition of weight between two fuzzy sets was put forward, and the fuzzy combination rule was proposed on the basis of the new weight. Compared with other generalizing combination rules, the numerical experiments and the results show that the new fuzzy combination rule in this paper can acquire more changing information to the change of fuzzy focal elements more effectively, and the combination results are not influenced by the critical point. It overcomes the insufficiencies of other existing combination rules and enhances the robustness of fusion decision systems effectively.

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