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Study on Piecewise Smooth Signal Reconstruction Algorithms based on Compressed Sensing

***Abstract**. In this paper, the unknown piecewise smooth signal was chosen as tested signal. After random matrix were chosen as measure matrix, we design the CS (Compressed Sensing) model for the unknown piecewise smooth signal. The signal was reconstructed using the OMP (Orthogonal Matching Pursuit) algorithm. The linear combination wavelet bases were proposed by the authors and were chosen as the sparse base in the CS model. The simulation results show that CS model by this paper can acquire the better approximation of the original signal.*

***Streszczenie.** W artykule opisano metodę rekonstrukcji sygnału odcinkowo-gładkiego, z wykorzystaniem algorytmu OMP. Dane uzyskane z modelu próbkowania oszczędnego (ang. Compressed Sensing) sygnału, umieszczono w wygenerowanej losowo macierzy pomiarowej. W algorytmie próbkowania wykorzystano falkowe kombinacje liniowe. Wykazano, że zastosowany model próbkowania oszczędnego pozwala na lepszą aproksymację sygnału. (**Analiza algorytmów rekonstrukcji sygnału odcinkowo-gładkiego – próbkowanie oszczędne**).*

Keywords: compressed sensing, line combination wavelet base, Orthogonal Matching Pursuit.
Słowa kluczowe: próbkowanie oszczędne, falkowa kombinacja liniowa, OMP.

1. Introduction

A theory called compressed sensing (CS) was proposed by Candes, Tao and Romberg [1-3], and Donoho [4]. The theory showed that a signal having a sparse or compressible representation in one basis can be recovered from projections onto a small set of measurement vectors that are incoherent with the sparsity basis. The number of measurements is far smaller than that required by traditional measurements, which is limited by Nyquist sampling theorem. The signal sparse representation is the fundamental premise of CS implementation, so the optimal sparse bases of the signals were widely researched [5-9]. Considered for signals and images that are sparse in a wavelet basis, the wavelet tree structure has been employed in non-statistical CS inversion[10]. For some of the current researches, the signal usually is known, and researches were mainly focused on signal sparse representations. In fact, CS Applications for the unknown signal are more important.

In this paper, the unknown piecewise smooth signal was chosen as tested signal. This article is organized as follows. The compressed sensing theory are discussed in Section 2. The wavelet base combination and the linear combination wavelet basis proposed in Section 3. After random matrix were chosen as measure matrix, we designed the CS models for the piecewise smooth signal. The signal was reconstructed using the OMP (orthogonal matching pursuit) algorithm in section 4. Finally, conclusions are given in Section 5.

2. Compressed Sensing Theory

Compressed sensing theory mainly includes sparse representation, measurement and reconstruction, and its implement scheme was shown in Fig.1.

2.1 Sparse representation

Consider a length-N, real-valued, one-dimensional , discrete-time signal f indexed as $f(n)$, $n \in [1,2,\cdots,N]$. According to the signal processing theory, the signal x is a linear combination of the basis (sparse basis) $\Psi^T = [\psi_1, \psi_2, \cdots, \psi_N]$, that is

(1) $$f = \sum_{k=1}^{N} \psi_k \alpha_k = \Psi \alpha$$

where $\alpha_k = \langle f, \psi_k \rangle$, α and f is $N \times 1$ column vector, and the sparse basis matrix Ψ is $N \times N$ with the basis vectors ψ_i as columns.

2.2 Measurement

In CS, we do not measure or encode the K significant α_k directly. Rather ,we measure and encode $M \prec N$ projections of the signal using the measure matrix $\Phi = [\varphi_1, \varphi_2, \cdots \varphi_M]$, where Φ_M^T denotes the transpose of Φ_M .

(2) $$y = \Phi f$$

where y is an $M\times 1$ column vector, and the measurement basis matrix Φ is $M\times N$. Then, by substituting f in equation (2) using equation (1), y can be written as

(3) $$y = \Phi f = \Phi \Psi \alpha = \Theta \alpha$$

where $\Theta = \Phi\Psi$ is $M\times N$ matrix. The measurement process is not adaptive, meaning that Φ is fixed and does not depend on the signal f . To ensure the stability of CS, the measurement matrix Φ must be incoherent with the sparsifying basis Ψ .

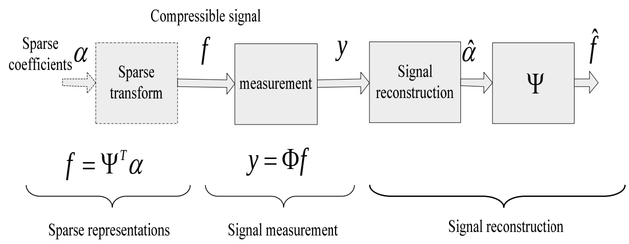

Fig.1 Signal processing scheme based on compressed sensing

2.3 Reconstruction

In Equation (3), there are N unknowns but only M equations, with $M << N$, the solutions for α are infinite. Obviously, it is an ill-posed problem recovering α from y. But according to the compressive sensing theory, if f is K-sparse and the condition $M \ge cK\log(N/K)$ is satisfied, we can exactly reconstruct the K-sparse vector with probality close to one by solving the following L_1 optimization:

(4) $$\min_{\alpha} \|\alpha\|_{L_1} \quad \text{subject to } y = \Phi\Psi\alpha$$

Because M is often much smaller than N, the recovery can be view as a sort of compression. The L_1 optimization problem in Equation 4 can be solved with linear programming methods. After we have solved α from the minimization problem, we can obtain *f* from Equation 1.

3 Wavelet Matrix

A widely used sparse representation in signal and image processing is the wavelet transform. As in compressive sensing, we assume f is sparse or nearly sparse in wavelet domain, i.e., $f = \Psi\alpha$, with Ψ corresponding to a wavelet matrix. In order to reconstruct f by solving an L_1 minimization problem as in Equation 4, we need to know the sensing matrix Θ, which is the multiplication of Φ and Ψ as shown by Equation 3. And if Ψ is known, can matrix Θ be calculated, then signal f can be reconstructed. Therefore, the sparse transform base Ψ is the key in compressed sensing.

3.1 Wavelet Base Combination

Because the piecewise smooth signal was not smooth in the whole region, the signal sparse representations cannot achieve better results using constant wavelet transform matrix. The jumps between two smooth parts of the signal generate large high frequency wavelet coefficients.

If the signal was not acquired, we can also understand some of its properties according to prior knowledge. For the piecewise smooth signal f, it can be written as the combination of the different region signal $f_1, f_2, \cdots, f_N$, so $f = f_1 + f_2 + \cdots + f_N$. For each smooth region f_i, according to the compressed sensing equation, we wrote as the followed equations.

(5) $$y_i = \phi_i f_i = \phi_i \psi_i \alpha_i = \theta_i \alpha_i,\ i = 1,2,\cdots,N$$

Based on compressed sensing model and the signal reconstruction algorithms, f_i can be calculated, so the piecewise smooth signal can be reconstructed.

3.2 Linear Combination Wavelet Base

Because we usually don't know the performances of the unknown signal, the linear combination wavelet bases which are more suitable for most applications had been proposed in this paper. The sparse representation for the signal *f* can be written the following equation by using linear combination wavelet basis.

(6) $$f = \Psi_c \alpha = (k_1\Psi_1 + k_2\Psi_2 + \cdots + k_N\Psi_N)\alpha$$

In order to keep energy conservation between the signal decomposition and reconstruction, the summation of all k_i coefficients is 1, that is

(7) $$k_1 + k_2 + \cdots + k_N = 1$$

Because the mixed base should present many different characteristics, the linear combination wavelet base must keep smooth, tight support, symmetry properties. For reducing the computing complexity, the linear combination bases usually choose 2-3 wavelet bases. When this linear combination wavelet base was used in CS theory, the CS model can be design as following equation.

(8) $$\begin{aligned} y &= \Phi\Psi_c\alpha = \Phi(k_1\Psi_1 + k_2\Psi_2 + \cdots + k_N\Psi_N)\alpha \\ &= k_1\Theta_1\alpha + k_2\Theta_2\alpha + \cdots + k_N\Theta_N\alpha \end{aligned}$$

After Ψ_c was calculated, we can reconstructed the signal using the OMP algorithms.

4. Experiment results

In order to study on sparse of wavelet bases combination and linear combination wavelet base proposed by this letter, the piecewise smooth signal which was shown in Fig.2 was chosen as the tested signal. For this test signal, we design three different length biorthogonal wavelet bases in table 1 for the different smooth regions. When the wavelet (1,1), wavelet (1,3) and wavelet (3,5) were chosen as the sparse base in the CS model respectively, the reconstruction signals using the orthogonal matching pursuit algorithms were shown in Fig.3, Fig.4 and Fig.5.

After acquiring the reconstruction signal using the single wavelet base listed in table 1, the above three wavelet base combination was chosen as the sparse transform base in the CS model. Because the smooth performances for each smooth regions of the tested signal are different, so wavelet (1,1) was used in the first smooth regions, wavelet (1,3) was used in second smooth regions, and wavelet (3,5) was used in last smooth regions. Using the OMP algorithms which were used in single wavelet CS model, the reconstruction signal was shown in Fig.6.

Because the signal properties usually are unknown, the sparse base should more generic. Therefore, the linear combination wavelet base was proposed in this paper. In order to meet demand of the comparative analysis, wavelet (1, 1), wavelet (1, 3) and wavelet (3, 5) were used in the linear combination wavelet base. According to Equation 13, K1=0.4, K2=0.4, k3=0.2 were chosen as the linear combination coefficients. When CS still adopted the OMP algorithms, the reconstruction signal was shown in Fig.7. In order to objectively evaluate the quality of reconstructed signals, the error can be calculated by the following equation.

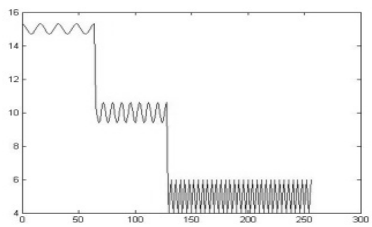

Fig.2 The original signal

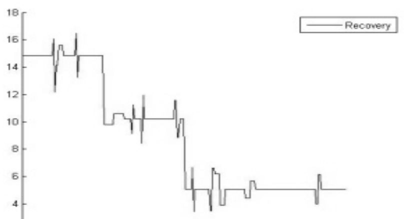

Fig.3 Reconstruction signal using wavelet (1, 1)

The reconstruction signal errors using the different wavelet bases were listed in Table 2.From the table 2 and the reconstruction signal using the different sparse bases, we can know that the reconstruct signal using the sparse basis proposed by this paper can acquired better approximation. The CS model using the linear combination wavelet base can also acquire better reconstruction signal and meet the demand of universal applications.

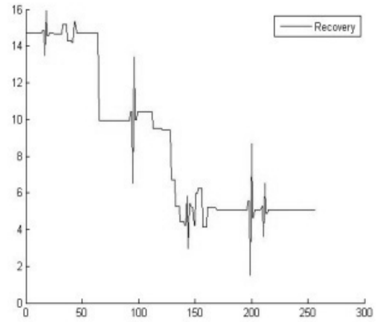

Fig.4 Reconstruction signal using wavelet (1, 3)

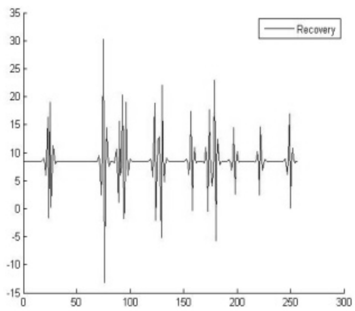

Fig.5 Reconstruction signal using wavelet (3, 5)

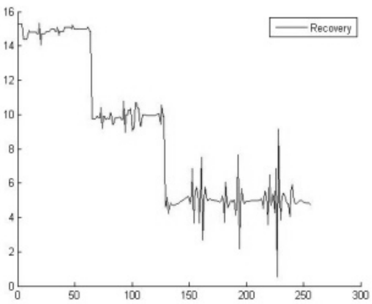

Fig.6 Reconstruction signal using wavelet combination

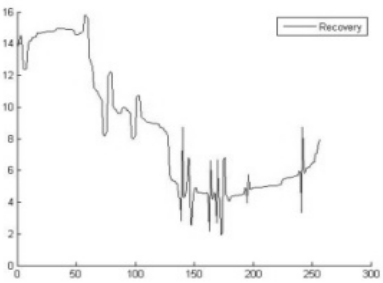

Fig.7 Reconstruction signal using linear combination wavelet

Table 1 Three different length biorthognormal wavelet

Wavelet Name	Analysis Filter Length	Synthesis Filter Length
Wavelet(1,1)	1	1
Wavelet(1,3)	1	3
Wavelet(3,5)	3	5

Table 2 Reconstruction signal error with using the different wavelets bases

Sparse base	error
Wavelet(1,1)	1.3240
Wavelet(1,3)	1.3173
Wavelet(3,5)	1.1466
Wavelet base combination	0.0776
Linear combination wavelet base	1.2634

Conclusion

The wavelet base combination and linear combination wavelet base were proposed in this letter. The simulation results show that the wavelet basis combination and linear combination wavelet base can acquire better reconstruction signal than that of single wavelet base. The wavelet base combination can get better error than that of the linear combination wavelet base, but linear combination wavelet base in CS model should meet demand of wide applications.

Acknowledgments

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