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Composite Barycentric Rational Interpolation with High-Accuracy

Abstract. *Rational interpolation gives much better approximations than polynomial interpolation, but it is difficult to avoid poles, unattainable points and control the occurrence of poles. In [1], a family of barycentric rational interpolants that have no poles and high approximation orders is given based on composite algorithm for the barycentric rational interpolation. In this paper, we propose a new composite barycentric rational interpolants with high-accuracy. The error estimation is discussed and a numerical example is given to show the effectiveness of our new method.*

treszczenie. *W artykule przedstawiono nowy, złożony i wymierny interpolator barycentryczny o wysokiej dokładności. W klasycznej formie interpolacja wymierna ma znacznie lepsze własności aproksymacji niż interpolacji wielomianowa, lecz trudno w niej jest uniknąć biegunów, punktów nieosiągalnych i sterować ich występowaniem. (**Złożona i wymierna interpolacja barycentryczna o wysokiej dokładności**).*

Keywords: barycentric rational interpolation; composite; accuracy.
Słowa kluczowe: wymierna interpolacja barycentryczna, kompozyt, dokładność.

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Introduction

The problem of interpolation is that according to the given values of discrete points to construct a simple continuously function such that it has the same function values at all the given points exactly. In this digitization age, it is not difficult to find examples of applications where this problem occurs. The relatively easiest and in many applications often most desired approach to solve the problem is interpolation, where an approximating function at the given measurement points. Polynomial interpolants are used in the solutions of equations and in the approximation of functions, of integral and differential equations, etc. Polynomial interpolants are used as the basic means of approximation. It is well known that the classical rational interpolation sometimes gives better approximations than polynomial interpolation[3]. But it is difficult to avoid and control poles and there are sometimes unattainable points and infinite inverse differences for the Thiele-type continued fraction interpolation[4]. Barycentric rational interpolation was presented by W. Werner[2], it possesses various advantages in comparison with classical continued fraction rational interpolants, such as small amount of calculation, good numerical stability, no poles and no unattainable points [2,5].

In [1], the composite barycentric rational interpolations have been given by Floater and Hormann. An important property of interpolants in [1] is that interpolants are free of poles. Furthermore, interpolants in [1] possess high-accuracy when the interpolants are smooth enough. However, this method has some deficiencies. For example, composite barycentric rational interpolation would degenerate into polynomial interpolation on the interval in extreme cases. And, the interpolant approximation accuracy is not very well obviously for large sequences of equidistant points. In this paper, we will adopt local barycentric rational interpolations to composite interpolants with high-accuracy. The interpolants obtained by new method possess good numerical stability, no poles and no unattainable points. At last, numerical examples are given to show the effectiveness of our new method.

Univariate Barycentric Rational Interpolation

Let the rational function $r(x) \in R_{n,n}$, $R_{n,n}$ is a set of all rational functions with the degrees of denominator and numerator at most n respectively .

Lemma [6] Let $\{(x_i, f_i), 0 \le i \le n\}$, be $n+1$ pairs of real numbers with $x_i \ne x_j$ for $i \ne j$, and let $f(x_i) = f_i$, and $\{\mu_i,\ 0 \le i \le n\}$ be $n+1$ pairs of real numbers. Then

a) If $\mu_k \ne 0$, the rational function $r(x)$ satisfy

(1) $$r(x) = \frac{p(x)}{q(x)} = \frac{\sum_{i=0}^{n} \frac{\mu_i}{x - x_i} f_i}{\sum_{i=0}^{n} \frac{\mu_i}{x - x_i}} \in R_{n,n}$$

and $\lim_{x \to x_k} r(x) = f_k$.

b) Conversely, every rational interpolantion $r(x) \in R_{n,n}$ may be written as in (1) for some μ_k .

According to lemma, when all weights of the interpolation nodes are not equal to zero, the barycentric rational interpolation has no unattainable points. From [5], the necessary condition of barycentric rational interpolation which has no poles is given as follows:

$$sign(\mu_j) = -sign(\mu_{j+1}), \quad (0 \le j \le n-1).$$

It is the key issue how to choose weights of barycentric rational interpolation to obtain a better approximation. Barycentric rational interpolants possess various advantages, such as, small amount of calculation[7] , good numerical stability, no poles and no unattainable points, and it is better than the polynomial interpolation interpolation when the interpolation points are equidistance[6] . In 1988, using the simple weights $u_j = (-1)^j \quad (0 \le j \le n)$, Berrut gave the barycentric rational interpolant as follows

(2) $$R(x) = \frac{\sum_{i=0}^{n} \frac{(-1)^i}{x - x_i} f_i}{\sum_{i=0}^{n} \frac{(-1)^i}{x - x_i}}.$$

This barycentric rational interpolation has no poles in the interpolation interval [8].

Composite Barycentric Rational Interpolation

Let $n+1$ distinct interpolation points $x_i\ (0 \le i \le n)$, together with corresponding numbers $f(x_i)$ be given, choose any integer $d\ (0 \le d \le n)$,and for each $i \le j \le i+d$ let $p_i(x)$ denote the unique polynomial of the degree at most d that interpolates $f(x)$ at the $d+1$ points

$x_i, x_{i+1}, \cdots, x_{i+d}$. Composite barycentric rational interpolation be given by Floater and Hormann in [1] as follows.

(3) $$R(x)=\frac{\sum_{i=0}^{n-d}\lambda_i(x)p_i(x)}{\sum_{i=0}^{n-d}\lambda_i(x)},$$

Where

(4) $$\lambda_i(x)=\frac{(-1)^i}{(x-x_i)(x-x_{i+1})\cdots(x-x_{i+d})},$$

for fixed $d \ge 1$ the interpolant has approximation order $O(h^{d+1})$ as $h \to 0$, where $h := \max_{0\le i\le n-1}(x_{i+1}-x_i)$, as long as $f(x)\in C^{d+2}$ [1].

However, this method has some deficiencies, for example, if you take $d=n$, the interpolant obtained by formula (3) will degenerate into polynomial interpolation on the interval. The interpolation approximation precision is not well obviously, especially for large sequences of equidistant points. Meanwhile, thanks to the barycentric rational interpolations which possess high-accuracy, rational interpolations with high-accuracy can be composited by barycentric rational interpolations.

Composite Barycentric Rational Interpolation With High-accuracy

Let $n+1$ distinct interpolation points $x_i\ (0\le i\le n)$ together with corresponding numbers $f(x_i)$ be given.

Choose any integer $d\ (0\le d\le n)$, and for each $i\le j\le i+d$, Let $r_i(x)$ denote the barycentric rational interpolations with $u_j=(-1)^j(0\le j\le n)$ that interpolate $f(x)$ at the $d+1$ points $x_i, x_{i+1}, \cdots, x_{i+d}$, then

(5) $$R(x)=\frac{\sum_{i=0}^{n-d}\lambda_i(x)r_i(x)}{\sum_{i=0}^{n-d}\lambda_i(x)},$$

where

(6) $$\lambda_i(x)=\frac{(-1)^i}{(x-x_i)(x-x_{i+1})\cdots(x-x_{i+d})},$$

(7) $$r_i(x)=\frac{\sum_{j=i}^{i+d}\frac{(-1)^j}{x-x_j}f_j}{\sum_{j=i}^{i+d}\frac{(-1)^j}{x-x_j}},\ i=0,1,\cdots,n-d.$$

Obviously, using formula (5), the drawback of formula (3) can be overcome effectively. When $d=n$ or $d=0$, the interpolant can still be barycentric rational interpolations on the whole interpolation interval. We can prove that the interpolants obtained by formula (5) satisfy interpolation conditions, have no poles. In the following, we will derive some properties of the interpolant and give some numerical examples in order to illustrate the new method with effectiveness.

Theorem 1 $R(x)$ obtained by formula (5) satisfy interpolation condition, $R(x_\alpha)=f(x_\alpha), \alpha=0\cdots n$.

Proof Let $J=\{i\in I, \alpha-d\le i\le \alpha\}\ (0\le\alpha\le n)$ $I=\{0,1,\cdots,n-d\}$, and $x=x_\alpha$. According to lemma, we can obtain $r_i(x_\alpha)=f(x_\alpha)\ (i\in J)$.

From $\mu_i(x_\alpha)\ne 0\ (i\in J)$, $\mu_i(x_\alpha)=0\ (i\in I\setminus J)$, and $\mu_i(x)=\prod_{j=0}^{i-1}(x-x_j)\prod_{k=i+d+1}^{n}(x_k-x)$, we obtain

$$R(x_\alpha)=\frac{\sum_{i=0}^{n-d}\lambda_i(x_\alpha)r_i(x_\alpha)}{\sum_{i=0}^{n-d}\lambda_i(x_\alpha)}=\frac{\sum_{i\in J}\mu_i(x_\alpha)r_i(x_\alpha)}{\sum_{i\in J}\mu_i(x_\alpha)}=\frac{\sum_{i\in J}\mu_i(x_\alpha)f(x_\alpha)}{\sum_{i\in J}\mu_i(x_\alpha)}$$

Theorem 2 For all $d(0\le d\le n)$, the rational interpolation $R(x)$ obtained by formula (5) has no poles in its interval.

Proof It follows from (5), we obtain[9]

$$R(x)=\frac{\sum_{i=0}^{n-d}\frac{(-1)^i}{(x-x_i)(x-x_{i+1})\cdots(x-x_{i+d})}\cdot\frac{\sum_{j=i}^{i+d}(-1)^j\prod_{k=i,k\ne j}^{i+d}(x-x_k)f_j}{\sum_{j=i}^{i+d}(-1)^j\prod_{k=i,k\ne j}^{i+d}(x-x_k)}}{\sum_{i=0}^{n-d}\frac{(-1)^i}{(x-x_i)(x-x_{i+1})\cdots(x-x_{i+d})}},$$

numerator and denominator of $R(x)$ are multiplied by $(-1)^{n-d}(x-x_0)\cdots(x-x_n)\cdot(\prod_{i=0}^{n-d}\sum_{j=i}^{i+d}(-1)^j\prod_{t=i,t\ne j}^{i+d}(x-x_t))$, then we have

$$R(x)=\frac{\sum_{i=0}^{n-d}(\mu_i(x)(\sum_{j=i}^{i+d}(-1)^j\prod_{k=i,k\ne j}^{i+d}(x-x_k)f_j))}{(\sum_{i=0}^{n-d}\mu_i(x))(\prod_{i=0}^{n-d}\sum_{j=i}^{i+d}(-1)^j\prod_{t=i,t\ne j}^{i+d}(x-x_t))}\cdot(\prod_{s=0}^{i-1}\sum_{j=s}^{s+d}(-1)^j\prod_{t=s,t\ne j}^{s+d}(x-x_t))(\prod_{s=i+1}^{n-d}\sum_{j=s}^{s+d}(-1)^j\prod_{t=s,t\ne j}^{s+d}(x-x_t)).$$

Let $s(x)=(\sum_{i=0}^{n-d}\mu_i(x))(\prod_{i=0}^{n-d}\sum_{j=i}^{i+d}(-1)^j\prod_{t=i,t\ne j}^{i+d}(x-x_t))$,

and $s_1(x)=\sum_{i=0}^{n-d}\mu_i(x)$. Floater and Hormann have proved that $\sum_{j=i}^{i+d}(-1)^j\prod_{t=i,t\ne j}^{i+d}(x-x_t)\ne 0$, for $u_j=(-1)^j$, then $\prod_{i=0}^{n-d}\sum_{j=i}^{i+d}(-1)^j\prod_{t=i,t\ne j}^{i+d}(x-x_t)\ne 0$ [8]. Meanwhile,we have already known that $s_1(x)=\sum_{i=0}^{n-d}\mu_i(x)>0$ [1], then, $s(x)\ne 0$. Thus, Theorem 2 was established.

Approximation Error

Theorem 3. Suppose $f(x)\in c^2[a,b]$, the rational interpolant $R(x)$ is obtained by formula (5), then if d is odd

(8) $$\|f(x)-R(x)\|\le h(1+\beta)\gamma\frac{\|f''\|}{2},$$

if d is even

(9) $$\|f(x)-R(x)\|\le h(1+\beta)[\gamma\frac{\|f''\|}{2}+\|f'\|],$$ where,

$$h=\max_{0\le i\le n-1}\{x_{i+1}-x_i\}, \gamma=\max_{1\le i\le n-d}\{x_{i+d}-x_i\},$$

$$\beta=\max_{1\le i\le n-2}\min\left\{\frac{x_{i+1}-x_i}{x_i-x_{i-1}}, \frac{x_{i+1}-x_i}{x_{i+2}-x_{i+1}}\right\}.$$

Proof If $d=0$, we can obtain the approximation error of the barycentric rational interpolation $r_i(x)$ with $u_j=(-1)^j \quad (0\le j\le n)$ [1]. For $i=0,1,\ldots,n-d$, if d is odd

$$\|f(x)-r_i(x)\|\le h_i\left(1+\beta_i\right)\left(x_{i+d}-x_i\right)\frac{\|f''\|}{2},$$

and, if d is even

$$\|f(x)-r_i(x)\|\le h_i\left(1+\beta_i\right)[(x_{i+d}-x_i)\frac{\|f''\|}{2}+\|f'\|].$$

On the other hand

$$\left|f(x)-R(x)\right|=\left|\frac{\sum_{i=0}^{n-d}\frac{(-1)^i}{(x-x_i)\ldots(x-x_{i+d})}\left(f(x)-r_i(x)\right)}{\sum_{i=0}^{n-d}\frac{(-1)^i}{(x-x_i)\ldots(x-x_{i+d})}}\right|.$$

Numerator and denominator of $\left|f(x)-R(x)\right|$ are multiplied by $(-1)^{n-d}(x-x_0)(x-x_1)\cdots(x-x_n)$, from [1], we obtain $\sum_{i=0}^{n-d}\mu_i(x)>0$, then if d is odd ,

$$\|f(x)-R(x)\|\le\max_{0\le i\le n-d}\left(h_i(1+\beta_i)(x_{i+d}-x_i)\frac{\|f''\|}{2}\right)$$

$$=h\left(1+\beta\right)\gamma\frac{\|f''\|}{2}\quad.$$

Similarly, if d is even

$$\|f(x)-R(x)\|\le h(1+\beta)\left(\gamma\frac{\|f''\|}{2}+\|f'\|\right).$$

The error analysis indicates that approximation error of the new method in this paper can be smaller than [1].

Numerical Examples

Example 1. We apply the method to $f(x)=e^{-x^2}$ firstly, for $x\in(-1,1)$, the points x_i are sampled uniformly by $x_i=-1+2i/n,\ (0\le i\le n)$. Interpolants can be obtained by formula (2), formula (3) and formula (5) (take $d=5$). The maximum absolute error is displayed as follows.

Table 1. Error Comparison

	N=100	N=200	N=300
Formula (2)	1.2029e-2	4.1180e-3	9.4225e-4
Formula (3)	2.3001e-4	2.1777e-3	4.3414e-4
Formula (5)	8.6228e-5	2.2153e-5	3.6450e-6

Example 2. We apply the method to $f(x)=e^x\sin(2x)$ for $x\in(-1,1)$, $d=4$, the points x_i are sampled uniformly by $x_i=-1+2i/100,\ (0\le i\le 100)$.

The maximum absolute error at those points are 2.7418×10^{-3}, 2.6738×10^{-2} and 1.9757×10^{-1}, obtained by new method ,formula (2) and formula (3), respectively.

The figure of the function interpolated and interpolant by using MATLAB7.1 .

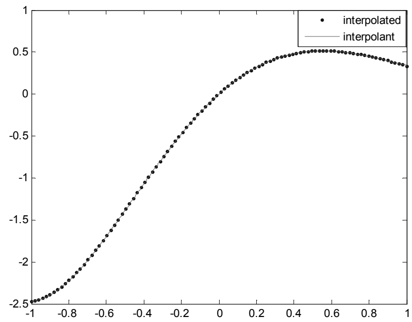

Fig.1. The interpolant with the new method.

Conclusion

In this paper, we have adopted local barycentric rational interpolations to composite interpolants with high-accuracy. The interpolants obtained by new method possess good numerical stability, no poles and no unattainable points. It can improve approximation accuracy forward by adopting local polynomial interpolation into barycentric rational interpolation, especially for large sequences of equidistant points.

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