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Magnetic field analysis nearby fractures of magnetic medium by means of BEM

***Abstract**. Magnetic field nearby fractures of magnetic medium is considered in this paper. Small thickness of fractures requires some kind of special treatment, which is a modified version of the boundary integral equation. It is solved then by means of the boundary element method (BEM).*

***Streszczenie.** W pracy rozważa się pole magnetyczne w otoczeniu pęknięć magnetyku. Z uwagi na założoną niewielką grubość pęknięć stosuje się graniczną postać równania całkowego, które rozwiązuje się dalej za pomocą metody elementów brzegowych. (**Analiza pola magnetycznego w otoczeniu pęknięć ośrodka magnetycznego za pomocą MEB**).*

Keywords: boundary element method, fractures, magnetic field.
Słowa kluczowe: metoda elementów brzegowych, pęknięcia, pole magnetyczne.

Introduction

Some real configurations contain thin bodies, like shells, fractures, gaps. Numerical analysis of field in neighborhood of such structures is usually very troublesome. In FEM, for example, a very fine mesh is required. It seems that it could be avoided at least in some class of problems, where use of BEM would be more suitable. In BEM, however, another problem arises – the occurrence of nearly-singular integrals, the numerical evaluation of which can be very inaccurate. There are at least three different methods of dealing with this problem. The first is to use special procedures to ensure high accuracy of the calculations, e.g [1]. The second method is to modify the boundary integral equation (BIE) so that the problem can be omitted [2-6]. Another approach is to use a different method, or to predict an approximate solution in the thin body [7-8].

This paper uses the modified BIE in magnetic field analysis nearby non-magnetic fractures in magnetic medium. It is based on the model presented in [2], but the final approach is different. It is similar to the procedure presented in [6].

Problem description

Consider a magnetic medium Ω_0 with a thin non-magnetic fracture Ω_1 of thickness d (Fig. 1). Let the relative permeability of the magnetic medium and the fracture is $\mu_{r0} >> 1$ and μ_{r1}, respectively. The magnetic medium is assumed to be linear. Initially, the fracture is considered as having boundaries S_+ (top) and S_- (bottom), but the final equations will concern the substitute surface S. In the magnetic medium there exists an externally applied magnetic field $\boldsymbol{H}_e$. In static conditions no currents are present and the magnetic field is assumed to have a scalar magnetic potential φ_e such that $\boldsymbol{H}_e = -\nabla\varphi_e$. The aim is find out the magnetic field nearby the fracture.

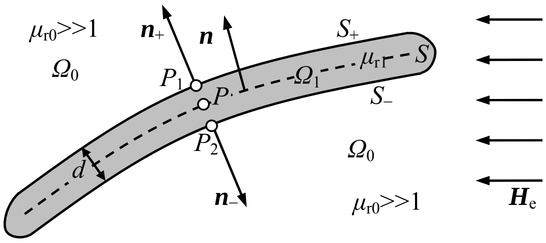

Fig.1. Fracture in magnetic medium

Governing equations

Provided that there are no currents (static magnetic field $\boldsymbol{H}_e$ or non-conductive medium at low frequencies of the externally applied field $\boldsymbol{H}_e$) one can use the scalar magnetic potential such that

$$\boldsymbol{H} = -\nabla\varphi . \tag{1}$$

In both regions Ω_0 and Ω_1, it satisfies the Laplace equation

$$\nabla^2\varphi^{(k)} = 0 \quad \text{in} \quad \Omega_k . \tag{2}$$

Magnetic field continuity on the boundaries S_+ and S_- leads to the following relationships

$$\varphi^{(0)} = \varphi^{(1)}, \qquad \mu_{r0}\frac{\partial\varphi^{(0)}}{\partial n} = \mu_{r1}\frac{\partial\varphi^{(1)}}{\partial n}, \qquad \text{on} \qquad S_{\pm} . \tag{3}$$

The externally applied field $\boldsymbol{H}_e$ can be introduced into the solution so that

$$\varphi^{(0)} \to \varphi_e \quad \text{at infinity.} \tag{4}$$

Boundary integral model

It can be shown that the initial BIE written for point i in Ω_0 with externally applied field has the following form [9]

$$\varphi_i^{(0)} + \int\limits_{S_+\cup S_-} \varphi^{(0)}\frac{\partial G}{\partial n_0}\mathrm{d}S = \int\limits_{S_+\cup S_-} G\frac{\partial\varphi^{(0)}}{\partial n_0}\mathrm{d}S + \varphi_{ei} , \tag{5}$$

where G is the fundamental solution for Laplace equation, and ∂_{n0} refers to normal derivative with normal $\boldsymbol{n}_0$ directed outwards Ω_0. Similar equation can be written for point i inside Ω_1:

$$\varphi_i^{(1)} + \int\limits_{S_+\cup S_-} \varphi^{(1)}\frac{\partial G}{\partial n_1}\mathrm{d}S = \int\limits_{S_+\cup S_-} G\frac{\partial\varphi^{(1)}}{\partial n_1}\mathrm{d}S . \tag{6}$$

If $d \to 0$, then G on S_+ as well S_- tends to G on S, the same for the normal derivatives of G, whereas values of $\varphi^{(k)}$ as well as $\partial\varphi^{(k)}/\partial n_k$ can differ much on both sides of S. Therefore, the equations become

$$\varphi_i^{(0)} - \int\limits_S \Delta\varphi^{(0)}\frac{\partial G}{\partial n}\mathrm{d}S = -\int\limits_S \Delta q^{(0)} G\,\mathrm{d}S + \varphi_{ei} , \tag{7}$$

$$(8)\qquad \int\limits_S \Delta\varphi^{(1)} \frac{\partial G}{\partial n}\,\mathrm{d}S = \int\limits_S \Delta q^{(1)} G\,\mathrm{d}S .$$

where $\partial/\partial n$ refers to the normal derivative of with normal vector $\boldsymbol{n}$ oriented from S_- to S_+ (Fig. 1), and

$$(9)\qquad \Delta\varphi^{(k)} = \varphi_+^{(k)} - \varphi_-^{(k)}, \qquad \Delta q^{(k)} = \frac{\partial\varphi_+^{(k)}}{\partial n} - \frac{\partial\varphi_-^{(k)}}{\partial n} ,$$

Symbols $\varphi_+^{(k)}$ and $\varphi_-^{(k)}$ denote the values of φ in domain Ω_k at boundaries S_+ and S_-, respectively. Note that there is no $\varphi_i^{(1)}$ in Eq. (8), because the term cancels with the singularities that are excluded from the boundary integrals for $d \to 0$. When point i lies on boundary, Eq. (7) also has a singularity. Its elimination leads to the following equation

$$(10)\qquad \begin{aligned} &(1-c_i)\varphi_{i+}^{(0)} + c_i\varphi_{i-}^{(0)} - \int\limits_S \Delta\varphi^{(0)} \frac{\partial G}{\partial n}\,\mathrm{d}S = \\ &\qquad = -\int\limits_S \Delta q^{(0)} G\,\mathrm{d}S + \varphi_{\mathrm{e}i}, \end{aligned}$$

where c_i is the geometric coefficient for point i lying on S ($^1/_2$ for smooth boundary).

Using relationships (3) in Eq. (8), one obtains

$$(11)\qquad \nu\int\limits_S \Delta\varphi^{(0)} \frac{\partial G}{\partial n}\,\mathrm{d}S = \int\limits_S \Delta q^{(0)} G\,\mathrm{d}S ,$$

where $\nu = \mu_{\mathrm{r}1}/\mu_{\mathrm{r}0}$. The relationship serves to eliminate the integral containing $\Delta q^{(0)}$ in Eq. (10) so that

$$(12)\qquad (1-c_i)\varphi_{i+} + c_i\varphi_{i-} + (\nu-1)\int\limits_S (\varphi_+ - \varphi_-)\,\partial_n G\,\mathrm{d}S = \varphi_{\mathrm{e}i} ,$$

where superscript (0) was omitted. Eq. (12) contains two unknown functions, φ_+ and φ_-. To find them out, an additional equation is required. It can be obtained by evaluating the gradient of Eq. (12) with respect to coordinates of point i, and taking its normal component [2]. As a result, $\partial_{ni}\varphi_{i\pm}$ appears in the equations. Due to small value of d the derivative can be approximated with a linear combination of φ_+ and φ_- as follows:

$$(13)\qquad \frac{\partial\varphi_\pm^{(0)}}{\partial n_i} = \frac{\mu_{\mathrm{r}1}}{\mu_{\mathrm{r}0}}\frac{\partial\varphi_\pm^{(1)}}{\partial n_i} \approx \nu\frac{\varphi_+^{(1)} - \varphi_-^{(1)}}{d} = \nu\frac{\varphi_+^{(0)} - \varphi_-^{(0)}}{d} .$$

Consequently, the lacking equation becomes

$$(14)\qquad \nu(\varphi_{i+} - \varphi_{i-}) + (\nu-1)d\int\limits_S (\varphi_+ - \varphi_-)\frac{\partial^2 G}{\partial n_i \partial n}\,\mathrm{d}S = d\frac{\partial\varphi_{\mathrm{e}i}}{\partial n_i} .$$

Discrete model (BEM)

Eqs. (12) and (14) can be solved in a BEM-like fashion [10-12]. For simplicity, elements with constant approximation of potential are used. The final equation system can be written as follows:

$$(15)\qquad \begin{bmatrix} (\mathbf{I}-\mathbf{c}) + (\nu-1)\hat{\mathbf{H}} & \mathbf{c} - (\nu-1)\hat{\mathbf{H}} \\ \nu\mathbf{I} + d(\nu-1)\tilde{\mathbf{H}} & -\nu\mathbf{I} - d(\nu-1)\tilde{\mathbf{H}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varphi}_+ \\ \boldsymbol{\varphi}_- \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\varphi}_\mathrm{e} \\ d\mathbf{D}_\mathrm{e} \end{Bmatrix} ,$$

where $\mathbf{c}$ is a diagonal matrix with diagonal elements c_i, $\boldsymbol{\varphi}_+$ and $\boldsymbol{\varphi}_-$ are vectors of nodal values of φ on S_+ and S_-, respectively, $\boldsymbol{\varphi}_\mathrm{e}$ and $\mathbf{D}_\mathrm{e}$ are vectors of nodal values of φ_e and $\partial_{ni}\varphi_\mathrm{e}$ at the boundary nodes, respectively, $\hat{\mathbf{H}}$ and $\tilde{\mathbf{H}}$ are matrices the elements of which are

$$(16)\qquad \hat{H}_{ij} = \int\limits_{S_j} \frac{\partial G(\mathbf{r},\mathbf{r}_i)}{\partial n}\,\mathrm{d}S, \qquad \tilde{H}_{ij} = \int\limits_{S_j} \frac{\partial^2 G(\mathbf{r},\mathbf{r}_i)}{\partial n_i \partial n}\,\mathrm{d}S .$$

Equation system (17) does not contain the normal derivative and has half the number of equations present in the conventional BEM. Instead, integrals $\tilde{H}_{ij}$ appear. One disadvantage is, however, that Eq. (13) assumes a constant approximation of normal derivative of φ inside the fracture. Such an assumption seems to be justified for very small values of d. However, one of the consequences of the assumption is that the normal components of magnetic flux density on S_+ and S_- are equal. This occurs if the magnetic field lines pass the fracture perpendicularly to S_+ and S_-. Such a situation takes place when $\mu_{\mathrm{r}0} >> \mu_{\mathrm{r}1}$ ($\nu \to 0$). Numerical simulations confirm this observation.

Numerical examples

The computational model was implemented in Mathematica 7.0. The first example is a straight fracture of length L, perpendicular to a uniform magnetic field ($\varphi_\mathrm{e} = H_0 x$). Its thickness is set to d = 0.01L and ν = 0 ($\mu_{\mathrm{r}0} >> \mu_{\mathrm{r}1}$). The fracture was divided into 20 boundary elements. Fig. 2 shows the field image, and Fig. 3 shows the values of magnetic potential on both sides of the fracture obtained with the modified (M) and conventional (C) BEM. The results are compared with analytical solution by the conformal mapping for d = 0. CBEM and MBEM results are similar in this case (due to analytical integration). However, numerical tests showed that CBEM fails for very small d, even with analytical integration.

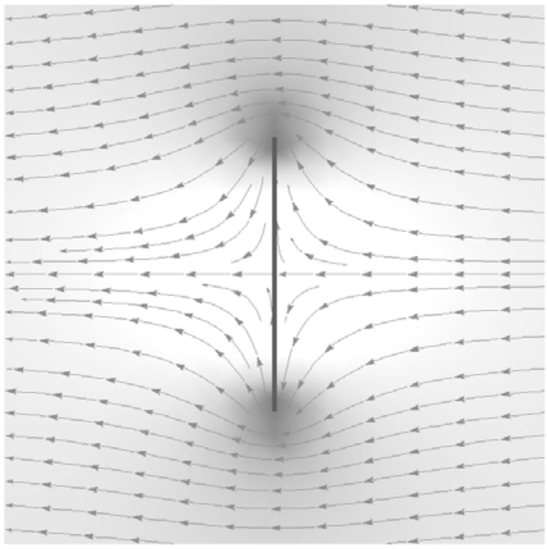

Fig.2. Magnetic field lines for $\nu \to 0$, d = 0.01L (shaded places indicate regions of stronger field)

Table 1 shows values of the relative magnetic flux through the fracture Φ_ν/Φ_1 for different d and ν, where

$$(17)\qquad \Phi_\nu = \mu_0\int\limits_S \frac{\partial\varphi^{(1)}}{\partial n}\,\mathrm{d}S \approx \mu_0\sum_j \frac{\varphi_{j+} - \varphi_{j-}}{d} S_j .$$

The flux depends roughly only on ratio $d/L/\nu$.

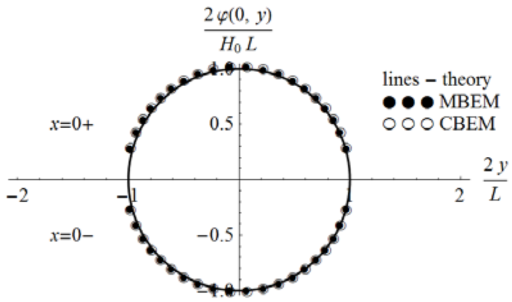

Fig.3. Values of scalar magnetic potential for perpendicular fracture and ν = 0 – theory, MBEM and CBEM

Fig. 4 shows field images for the straight and perpendicular fracture for different ν and constant d = 0.01L.

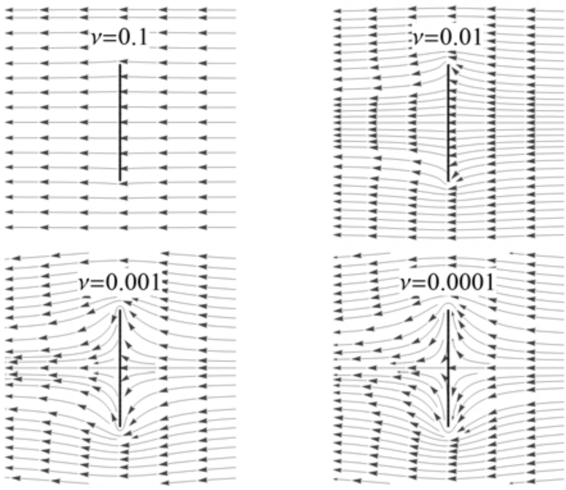

Fig.4. Magnetic field lines through straight perpendicular fracture (d = 0.01L and different ν)

Table 1. Relative magnetic flux Φ_ν/Φ_1 through perpendicular fracture

	d/L = 0.01	d/L = 0.001	d/L = 0.0001
ν = 0.1	0.932	0.992	0.999
ν = 0.01	0.599	0.926	0.991
ν = 0.001	0.137	0.597	0.925
ν = 0.0001	0.016	0.137	0.597

The next figures show exemplary field images nearby curved fractures (Fig. 5) and a set of straight fractures occurring randomly in the magnetic medium (Fig. 6).

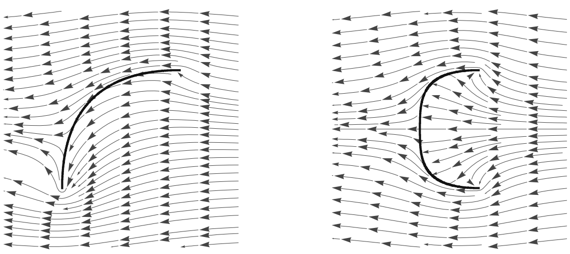

Fig.5. Magnetic field lines for some curved fractures (d about 0.01 of fracture length, ν = 0.001)

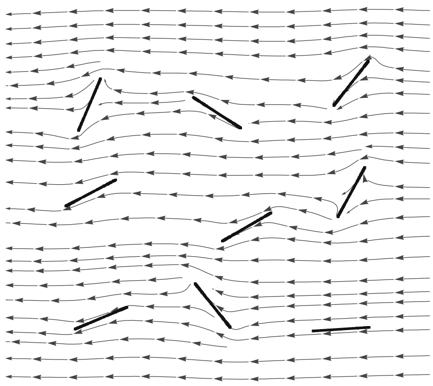

Fig.6. Magnetic field lines nearby a set of random fractures (ν = 0.001, d = 0.01 of fracture length)

Concluding remarks

The modified BEM allows the thin fractures in magnetic media to be modeled easily. When compared to the conventional BEM, it has the following advantages:

- half the number of equations,
- no nearly singular integrals,
- the thinner the fracture the better approximation,

The main disadvantage is that it uses a linear approximation for the normal derivatives on the fracture surface (see Eq. (13)). Theoretical considerations and numerical tests showed that this assumption can be hold if either ν << 1, or the fracture is placed perpendicularly to the magnetic lines of the externally applied field. Despite this limitations the presented model seems to work well in the considered class o problems.

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