Paweł JABŁOŃSKI

Częstochowa University of Technology

Use of BEM in electroconductive field analysis nearby thin highly conductive bodies

***Abstract**. The analysis of electroconductive field nearby thin highly conductive bodies is considered in this paper. The model uses the boundary element method (BEM), but it simplifies the analysis by using a limiting version of the boundary integral equation. The model can be used in the analysis of current density distribution in electrolyte nearby thin electrodes or metallic sheets.*

***Streszczenie.** W artykule rozpatruje się sposób analizy pola przepływowego w pobliżu cienkich dobrze przewodzących warstw. Model wykorzystuje metodę elementów brzegowych (MEB), ale upraszcza analizę poprzez zastosowanie granicznej postaci równania całkowo-brzegowego. Można go zastosować na przykład do analizy rozkładu gęstości prądu w elektrolicie w otoczeniu cienkich elektrod lub blach. (**Zastosowanie MEB do analizy pola przepływowego w pobliżu cienkich dobrze przewodzących ciał**).*

Keywords: thin bodies, good conductors, electroconductive field, BEM.
Słowa kluczowe: cienkie warstwy, przewodniki, pole przepływowe, MEB.

Introduction

Many real configurations contain bodies in the form of thin layers, shells or sheets. Examples are external shells of cars, planes, submarines, sheet metals, plates, some types of electrodes in electrolyte, equipment casings, etc. Such bodies are often called "thin bodies", and can be described as those whose one of the dimensions is much smaller than the others. Due to complexity of such configurations the analysis of electromagnetic field in them usually require a use of numerical methods. No matter which numerical method is to be used, thin bodies often require a special treatment. In FEM, for example, they generate a very fine mesh and result in a large system of equations. Since the problem is linear, BEM can be also used, especially if the domain is theoretically unbounded. In the conventional BEM, however, the problem of accuracy in evaluation of so called nearly singular integrals appears. It originates from the form of the coefficients in the resulting algebraic system of equations. They are integrals the integrands of which are functions of the negative power of distance between the source point and observation point lying in the boundary. If the source point is very close to the boundary surface, the integrand value changes very rapidly and numerical evaluation of such integral can be very inaccurate. This is not a problem when the integral can be done analytically, but often the integrals can be done only numerically.

The way out is to transform the governing equations to obtain more applicable ones. Some of approaches are considered in [1-7]. The proposed method is based on the suitably adopted model described in [1, 2]. It uses the boundary integral equation (BIE) corresponding to the problem, but the thin body, which is assumed to be a good conductor, is replaced by a single surface (line in 2D case). This allows simplifying considerably the resulting equation and eliminates the nearly singular integrals. In addition, the number of equations in the final system of algebraic equations is reduced.

Problem description and the governing equations

A thin plate Ω_1, of thickness d, is placed in conductive medium Ω_0 (Fig. 1). The conductivity of the medium and the plate equals γ_0 and γ_1, respectively, with the additional assumption $\gamma_1 >> \gamma_0$. This case corresponds to metallic plates placed in conductive medium, for example. Other cases, e.g. $\gamma_1 << \gamma_0$, can be also taken into account, e.g. in [5, 7]. The assumptions allow the subsequent equations to be simplified considerably. The whole configuration is affected by an externally applied static electric field $\boldsymbol{E}_s$.

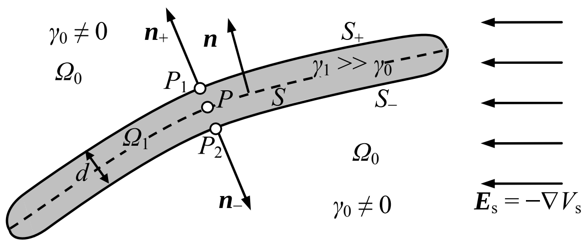

Fig.1. Thin plate in conductive medium

Initially, the thin plate is modelled as a conventional body with its boundary surface consisting of surfaces S_+ and S_-, but the final equations will be formulated for a substitutive surface S.

The scalar potential V satisfies the Laplace equation in domain Ω_0

(1) $$\nabla^2 V = 0 \quad \text{in} \quad \Omega_0 .$$

Since the plate is assumed to be highly conductive, its potential is approximately constant and equal U so that

(2) $$V\big|_{S_\pm} = U .$$

The externally applied field $\boldsymbol{E}_s$ has potential V_s (i.e. $\boldsymbol{E}_s = -\nabla V_s$), and far from the plate, which disturbs the $\boldsymbol{E}_s$ field, potential V should be equal V_s:

(3) $$V\big|_\infty \to V_s .$$

In this model U can be known or not. If it is known, no further equations are required to obtain the solution. If U is unknown, however, one more equation is needed. It can be provided by the equation of continuity of current, which yields

(4) $$\int_{S_+} \frac{\partial V}{\partial n_+} \mathrm{d}S + \int_{S_-} \frac{\partial V}{\partial n_-} \mathrm{d}S = -\frac{I}{\gamma_0} ,$$

where I is the current provided into the plate by external agents. For example, if the plate is an electrode, it introduces current I into the conductive medium Ω_0. The current is negative, if the plate gathers the current from the medium. For a free plate, one should assume I = 0, and the value of U is to be found. For 2D problems I is the current per unit of length of the electrode.

The BIE corresponding to Eqs. (1)-(3) written for point i lying in domain Ω_0 is as follows:

(5) $$V_i + \int_{S_+} G \frac{\partial V}{\partial n_+} \mathrm{d}S + \int_{S_-} G \frac{\partial V}{\partial n_-} \mathrm{d}S = V_{\mathrm{s}i} ,$$

where V_i − the value of V at point i, V_{si} – the value of V_{s} at point i, and G is the fundamental solution for the Laplace equation. Two facts are worth noting: (i) there is no integral with $V\partial G/\partial n$, because it vanishes due to condition (2); (ii) signs of the integrals of $G\partial V/\partial n$ come form the inner direction of normal vectors $\boldsymbol{n}_+$ and $\boldsymbol{n}_-$. It can be shown that Eq. (5) is correct also for point i lying on the boundary of the plate.

Conventional BEM model (CBEM)

Eqs. (5) and (4) can be solved by means of BEM. If each of boundaries S_+ and S_- is divided into N zero-order boundary elements, then a total of $2N$ + 1 equations will be obtained. Assuming known I, the equations are as follows:

(6) $$\begin{bmatrix} \mathbf{G}_{++} & \mathbf{G}_{+-} & \mathbf{1} \\ \mathbf{G}_{-+} & \mathbf{G}_{--} & \mathbf{1} \\ \mathbf{s}_+ & \mathbf{s}_- & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{Q}_+ \\ \mathbf{Q}_- \\ U \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}_{\mathrm{s}+} \\ \mathbf{V}_{\mathrm{s}-} \\ -I/\gamma_0 \end{Bmatrix} ,$$

where $\mathbf{G}_{kl}$ – G-type BEM matrix corresponding to points i on boundary S_k and boundary elements on boundary S_l, $\mathbf{Q}_k$ and $\mathbf{V}_{\mathrm{s}k}$ – column vectors of values of $\partial V/\partial n_k$ and V_{s} in nodes on boundary S_k, respectively, s_k − row vectors of lengths of boundary elements on boundary S_k, $\mathbf{1}$ – column vector of N ones. Elements of matrices $\mathbf{G}_{kl}$ are integrals

(7) $$\int_{S_j} G(\boldsymbol{r},\boldsymbol{r}_i)\,\mathrm{d}\boldsymbol{S} .$$

Although Eq. (6) is mathematically correct, during numerical evaluation two problems arise for thin enough plate: (i) some of integrals (7) become nearly singular, and therefore, hard for numerical evaluation with sufficient accuracy (e.g. for curved boundary elements); (ii) the system of equations becomes badly conditioned, because all four matrices $\mathbf{G}_{kl}$ are almost the same. Numerical tests confirm that the model crashes for sufficiently small d.

Modified BIE and its BEM model (MBEM)

If $d \to 0$, boundary $S_+ \to S$, and $S_- \to S$, too, but it does not mean that $\partial V/\partial n_+$ and $\partial V/\partial n_-$ become the same (with the opposite sign). In fact, the values can differ much on both sides of the plate. All we can do is to use a substitute surface S instead of real plate with surfaces S_+ and S_-, and then make the following approximations:

(8) $$\left.\frac{\partial V}{\partial n_+}\right|_{S_+} \approx \left.\frac{\partial V_+}{\partial n}\right|_S , \qquad \left.\frac{\partial V}{\partial n_-}\right|_{S_-} \approx -\left.\frac{\partial V_-}{\partial n}\right|_S .$$

As for the fundamental solution G, it depends only on the distance between the observation and source points, and therefore for $d \to 0$

(9) $$G\big|_{S_+} \to G\big|_S \leftarrow G\big|_{S_-} .$$

Inserting Eqs. (8) and (9) into Eq. (5) yields the following modified BIE

(10) $$V_i + \int_S \Delta q G \,\mathrm{d}S = V_{\mathrm{s}i} ,$$

where

(11) $$\Delta q = \left.\frac{\partial V_+}{\partial n}\right|_S - \left.\frac{\partial V_-}{\partial n}\right|_S .$$

In such a way the real plate has been led to a substitute surface S. Eqs. (11) and (4) can be solved by means of BEM. If surface S is divided into N zero-order boundary elements, then a total of N + 1 equations will be obtained as follows:

(12) $$\begin{bmatrix} \mathbf{G} & \mathbf{1} \\ \mathbf{s} & 0 \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{q} \\ U \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}_{\mathrm{s}} \\ -I/\gamma_0 \end{Bmatrix} ,$$

On the other hand, if U is known, then the system of equations takes the following form:

(13) $$\begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{s} & 1 \end{bmatrix} \begin{Bmatrix} \Delta\mathbf{q} \\ I/\gamma_0 \end{Bmatrix} = \begin{Bmatrix} \mathbf{V}_{\mathrm{s}} - U\mathbf{1} \\ 0 \end{Bmatrix} .$$

In both cases the main problems with small value of d have been removed. In addition, the discretization is required only to the substitutive surface, what results in easier preprocessing as well as in smaller system of equations in comparison to the conventional model given by Eq. (6). Its disadvantage is that it uses approximations valid for $d \to 0$ and $\gamma_1 \to \infty$.

Generalizations

The proposed model can be easily extended and generalized. Possible generalizations are:
- multiple highly conductive plates,
- including low conductive plates, like fractures and gaps, according to the model presented in [7],
- building a mixed body problem model, with non-thin bodies treated conventionally,
- formulation for bounded regions,
- analysis of other fields.

As for the last point, the model should be appropriate for any field $\boldsymbol{F}$ described by equations:

(14) $$\nabla \times \boldsymbol{F} = 0, \qquad \boldsymbol{F} = -\nabla V, \qquad \nabla \cdot (p\boldsymbol{F}) = 0 ,$$

where p is the material coefficient. Some exemplary fields are shown in Table 1.

Table 1. Examples of configurations for which the presented model can be used

Case	$\boldsymbol{F}$ field	V field	p parameter	U value	I value
Perfect conductor in electroconductive field	Electric field intensity	Scalar electric potential	Electrical conductivity	Scalar electric potential of the plate	Total current from the plate
Conductor in electrostatic field	Electric field intensity	Scalar electric potential	Relative permittivity	Scalar electric potential of the plate	Total charge on the plate
Highly permeable plate in static magnetic field	Magnetic field intensity	Scalar magnetic potential	Relative permeability	Scalar magnetic potential of the plate	Total magnetic flux form the plate (0)
Heat conductor in static temperature field	Negative gradient of temperature	Temperature	Thermal conductivity	Temperature of the plate	Total heat flux from the plate

Numerical examples

The presented model was implemented in *Mathematica* 7.0 and tested on several benchmark problems. The first one is a long flat highly conductive thin bar placed in an externally applied uniform electric field. This problem has an exact solution which can be found by means of conformal mapping. Fig. 2 shows the plot of distribution of Δq along the cross-section. The MBEM gives quite accurate results, except for the endpoints. The CBEM model gives almost the same results as MBEM, but it crashes for very small d. Fig. 3 shows the field image found be MBEM.

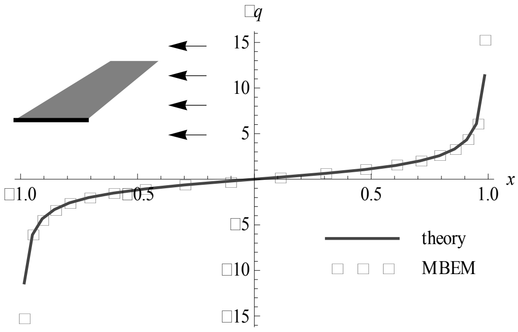

Fig.2. Values of Δq for flat conductive bar in uniform electric field

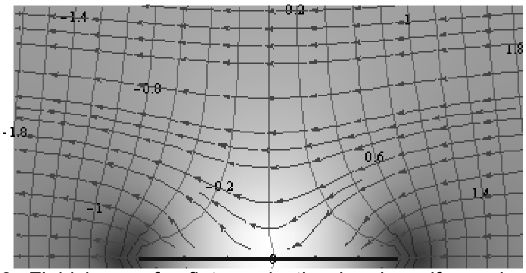

Fig.3. Field image for flat conductive bar in uniform electric field (cross-section of upper half)

The second benchmark problem is the same bar of known potential placed in conductive medium without externally applied electric field (Fig. 4 and 5). Again the conformal mapping allows finding the exact solution. The results are similar to those of the first example.

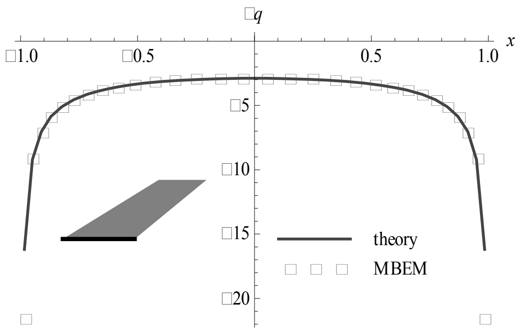

Fig.4. Values of Δq for flat conductive bar with given potential

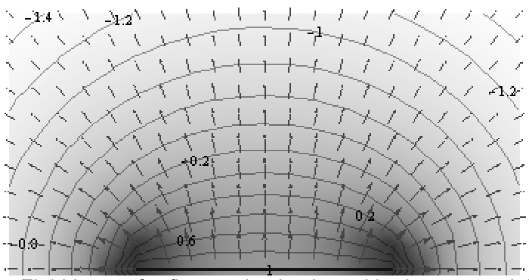

Fig.5. Field image for flat conductive bar with given potential (upper half cross-section)

The last example shows two electrodes in electrolyte. The field image (MBEM) is shown in Fig. 6.

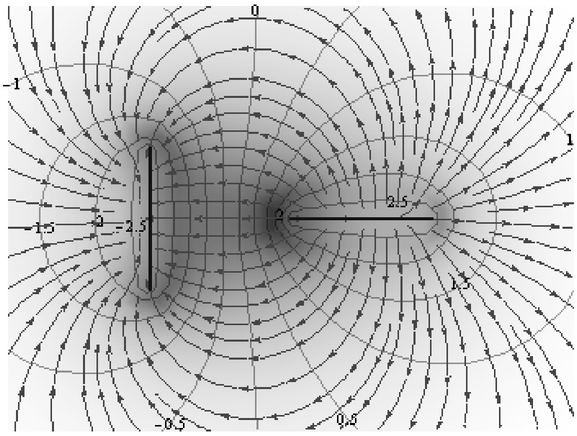

Fig.6. Field image for two long and thin electrodes in electrolyte (cross-section)

Concluding remarks

An approximate method of analysis of electroconductive field nearby thin highly conductive bodies was presented. The model can be easy adapted for other static fields, e.g. for thin conductive bodies in electrostatic field, or thin magnetic bodies in static magnetic field. When compared to the conventional BEM approach, it has the following advantages:

- no nearly singular integrals (for sufficiently regular shapes of the thin bodies),
- avoiding the crash due to badly conditioned CBEM matrix for very thin bodies,
- smaller system of equations resulting in much faster computations.

The model can be generalized to obtain a broader class of problems, i.e. multiple thin bodies, thin bodies of low conductivity.

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Author: *dr inż. Paweł Jabłoński, Częstochowa University of Technology, Electrical Faculty, al. Armii Krajowej 17, 42-200 Częstochowa, E-mail: paweljablonski7@gmail.com.*