Exact closed form formula for mutual inductance of conductors of rectangular cross section

Abstract. In this paper, using a definition of a mutual inductance for two conductors of any shape and lengths, the new exact closed formula for mutual inductance between two rectangular bars of any lengths is proposed. In the case of direct current (DC) or low frequency (LF) this inductance is given by analytical formula. The mutual inductance between two thin tapes of any lengths is also presented.

Introduction

The circuit system lumped conductors can be modeled by a connection of resistances, self and mutual inductances. The self and mutual inductances play an important role not only in power circuits, but also in printed circuit board (PCB) lands [1-8]. Formulae for the mutual inductances of set of conductors of rectangular cross-section are the subjects of many electrical papers and books. They are mathematically complex and their demonstrations are usually omitted and many electrical papers and books. They are mathematically complex and their demonstrations are usually omitted and therefore are presented in several well-known ones given in the literature for DC, low frequency or parallel thin tapes.

We consider a general case of two parallel conductors of rectangular cross section shown in Fig.1. The positions of conductors are determined by coordinates of diagonal corner points: \((s_1, s_6, s_9), (s_2, s_8, s_10)\) of the first wire of dimensions \(a_1 \times b_1 \times l_1\) and \((s_3, s_7, s_11), (s_4, s_8, s_12)\) of the second wire of dimensions \(a_2 \times b_2 \times l_2\). We assume the conductors to be parallel.

Definition of mutual inductance

The definition of mutual impedance between two straight conductors is given in \([17, 18]\) i.a. by following formula

\[
Z_{12} = \frac{j \omega \mu_0}{4\pi} \int_{l_1} \int_{l_2} \frac{J_{22}(Y) J_{11}^*(X)}{\rho_{xy}} \, d\nu_1 \, d\nu_2
\]

where \(J_{22}(Y)\) is the complex current density at source point \(Y = (x_1, y_1, z_1) \in S_2\), \(J_{11}^*(X)\) is the complex conjugate current density at point of observation \(X = (x_1, y_1, z_1) \in S_1\), \(\nu_1\) and \(\nu_2\) are conductors’ volumes. Distance between the point of observation \(X\) and the source point \(Y\) (Fig.1) is

\[
\rho_{xy} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.
\]

If conductors have a constant cross-sectional area \(S_1\) and \(S_2\) along theirs lengths, in case of DC, low frequency (for busbars of dimensions used in electrical power distribution system \([1]\)) or for a thin strip conductors (in printed circuit board \([2-8]\)) we can assume that the current density is constant and given as \(J_{11}(X) = I_{11} / S_1\) and \(J_{22}(Y) = I_{22} / S_2\) then, from the formula (1), we obtain the mutual inductance between two straight parallel conductors

\[
M = M_{12} = M_{21} = \frac{\mu_0}{4\pi} \int_{l_1} \int_{l_2} \frac{1}{\rho_{xy}^2} \, d\nu_1 \, d\nu_2
\]
Mutual inductance between parallel conductors of rectangular cross section

The mutual inductance between two rectangular conductors of dimensions \(a_1 \times b_1 \times l_1\) and \(a_2 \times b_2 \times l_2\) shown in Fig. 1 is given by formula

\[
M = \frac{\mu_0}{4 \pi} \frac{1}{a_1 b_1} F
\]

where

\[
F = \int_{s_{1y}}^{s_{2y}} \int_{s_{1x}}^{s_{2x}} \frac{1}{\rho_{XY}} \, ds_1 ds_2
\]

is a sixtuple definite integral of an integrable function \(\rho_{XY}\) of six variables \((x_1, y_1, x_2, y_2, z_1, z_2)\).

In general case this integral is very difficult to calculate. But in our case we can put \(x = x_2 - x_1\), \(y = y_2 - y_1\), \(z = z_2 - z_1\) and first to calculate a sixtuple indefinite integral

\[
F(x, y, z) = \int_{y_{1}}^{y_{2}} \int_{x_{1}}^{x_{2}} \int_{z_{1}}^{z_{2}} \int_{z_{1}}^{z_{2}} \frac{1}{\rho_{XY}} \, dx dy dz dz
\]

of a function

\[
\rho_{XY}^{-1}(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}}
\]

of three variables \((x, y, z)\) - twice with respect to \(x\), twice with respect to \(y\) and twice with respect to \(z\). After each double integration we omit terms which depend only on one or two variables - they are constants with respect to the considered variable. We can also omit terms proportional to one variable like \(H(y, z, x) = x g(y, z)\).

Finally, after a lengthy integration, formula (4) yields an expression for sixtuple indefinite integral

\[
F(x, y, z) = \frac{6}{5} \left(\frac{x^4 + y^4 + z^4 - 3x^2y^2 - 3x^2z^2 - 3y^2z^2}{4(x^2 + y^2 + z^2)^{\frac{3}{2}}} \right)
\]

After each double integration we omit terms which depend only on one or two variables - they are constants with respect to the considered variable. We can also omit terms proportional to one variable like \(H(y, z, x) = x g(y, z)\).

The second step gives the function

\[
F(z) = \int_{z_{1}}^{z_{2}} F(x, y, z) \, dz = \sum_{i=1}^{m} (-1)^{i+1} F(p_i, q_j, z)
\]

And, finally we have

\[
F(z) = \int_{z_{1}}^{z_{2}} F(x, y, z) \, dz = \sum_{i=1}^{m} \sum_{j=1}^{n} (-1)^{i+j} F(p_i, q_j, r_k)
\]

Hence the mutual inductance between two parallel conductors of rectangular cross section is given by following formula

\[
M = \frac{\mu_0}{4 \pi} \frac{1}{a_1 b_1} \left[ \int_{y_{1}}^{y_{2}} \int_{x_{1}}^{x_{2}} \frac{\rho_{XY}}{\rho_{XY}} \, dy \, dx \right] = \frac{\mu_0}{4 \pi} \frac{1}{a_1 b_1} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} (-1)^{i+j+k} F(p_i, q_j, r_k)
\]

where \(a_1 = s_2 - s_1\), \(b_1 = s_6 - s_5\), \(a_2 = s_4 - s_3\), \(b_2 = s_8 - s_7\), \(l_1 = s_{10} - s_9\) and \(l_2 = s_{12} - s_{11}\).

Mutual inductance between two parallel thin tapes

The mutual inductance between two parallel thin tapes of widths \(a_1\) and \(a_2\), lengths \(l_1\) and \(l_2\) respectively, thickness \(b_1 \approx 0\) and \(b_2 \approx 0\) and distance \(d\) between them (Fig.2) is given by formula

\[
M = \frac{\mu_0}{4 \pi} \frac{1}{a_1} F
\]

where

\[
F = \int_{y_{1}}^{y_{2}} \int_{x_{1}}^{x_{2}} \int_{z_{1}}^{z_{2}} \frac{dxdydz}{d^2 + (x_2 - x_1)^2 + (z_2 - z_1)^2}
\]

is a quadruple definite integral of four variables \((x_1, x_2, z_1, z_2)\) into which the distance \(d = s_7 - s_4\) is measured from the plan of the first tape to the plan of the second one. Now we can put \(x = x_2 - x_1\) and \(z = z_2 - z_1\) and first to calculate a quadruple indefinite integral

\[
F(x, z) = \int_{x_{1}}^{x_{2}} \int_{z_{1}}^{z_{2}} \int_{z_{1}}^{z_{2}} \frac{dx dy dz}{d^2 + x^2 + z^2}
\]

twice with respect to \(x\) and twice with respect to \(z\). Finally, after a lengthy integration, formula (11) yields an expression for quadruple indefinite integral
Computational results

where

Hence the mutual inductance between two thin tapes is given by following formula

Table 1. Mutual inductance between two busbars of rectangular cross section for DC or low frequency.

<table>
<thead>
<tr>
<th>l (m)</th>
<th>Reuhl L (nH)</th>
<th>Grover L (nH)</th>
<th>Strusky L (nH)</th>
<th>Hoer L (nH)</th>
<th>Eq. (9) L (nH)</th>
<th>Eq. (10) L (nH)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>0.03999</td>
<td>0.03999</td>
<td>0.03999</td>
<td>0.03922</td>
<td>0.03922</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>3.92230</td>
<td>3.92230</td>
<td>3.92169</td>
<td>3.84978</td>
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<tr>
<td>10.0</td>
<td>238.821</td>
<td>238.821</td>
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<tr>
<td>100</td>
<td>5800.11</td>
<td>5800.11</td>
<td>5799.86</td>
<td>5769.18</td>
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<tr>
<td>1000</td>
<td>94556.0</td>
<td>94556.0</td>
<td>94553.5</td>
<td>94240.4</td>
<td>94240.4</td>
<td>94373.4</td>
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</tbody>
</table>

Table 2. Mutual inductance between two tapes of rectangular cross section for DC or low frequency.

<table>
<thead>
<tr>
<th>l (m)</th>
<th>Reuhl L (pH)</th>
<th>Grover L (pH)</th>
<th>Strusky L (pH)</th>
<th>Eq. (9) L (pH)</th>
<th>Eq. (14) L (pH)</th>
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<tr>
<td>0.10</td>
<td>0.00024</td>
<td>0.00024</td>
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<td>0.00024</td>
<td>0.00024</td>
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<tr>
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<td>0.02451</td>
<td>0.02451</td>
<td>0.02449</td>
<td>0.02408</td>
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<tr>
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<td>1.49263</td>
<td>1.49195</td>
<td>1.47715</td>
<td>1.47715</td>
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<tr>
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<td>36.2507</td>
<td>36.2425</td>
<td>36.0634</td>
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<tr>
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<td>590.975</td>
<td>590.975</td>
<td>590.892</td>
<td>588.003</td>
<td>588.010</td>
</tr>
</tbody>
</table>

For the chosen transversal dimensions and different lengths of a busbars the calculations of their mutual inductance have been made according to all previous, shown above, formulae — Table 1.

For the mutual inductance of thin tapes of width $a$, thickness $b$ and length $l$ above formulae gives results sown in Table 2.

Conclusions

We have defined the mutual inductance of two conductors of any shape and any length given by sixtuple definite integral. For rectangular conductors the limits of this integral are given by coordinates of diagonal corner points of the first conductor and the second one. In the case of DC or low frequency we have given general formulae for the mutual inductance of conductors of rectangular cross section of any dimensions including the thickness. These formulae can be used for any dimensions of conductors and for any position between them.

By computations we have shown that our formulae give the same results as first of all Hoer’s ones for all dimensions of conductors. In addition we have also obtained analytical forms for the mutual inductance between the thin tapes. Of course they give the same results as first of all Hoer’s ones for all dimensions of conductors and for any position between them.
analytically simple and can also replace the traditional tables and working ones.

These formulae can be used in the methods of numerical calculation of AC mutual inductance of rectangular conductors. Then the cross sections of the conductors are divided into rectangular subbars (elementary bars) in which the current is assumed to be uniformly distributed over the cross section of each subbar.

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REFERENCES


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