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A class of electrostatic problems involving a circular annulus

Abstract. *This paper deals with the distribution of electric scalar potential within an infinitely long electrodes system the cross section of which is a circular annulus. We provide guidelines about an efficient method for solving a wide class of electrostatic problems.*

Streszczenie. *W artykule omówiono zagadnienie rozkładu potencjału elektrycznego na nieskończenie długich elektrodach, modelujących pierścień kołowy. Przedstawiono sposoby rozwiązywania zagadnień elektrostatyki, dotyczącej danego przypadku. (**Zagadnienia elektrostatyczne dotyczące pierścieni kołowych**).*

Keywords: electric scalar potential, Laplace's and Poisson's equations, functions of circular annulus
Słowa kluczowe: potencjał elektryczny skalarny, równania Laplace'a i Poisson'a, funkcjonalność pierścienia kołowego.

Preliminaries

In this part of the paper we will remind the reader of the well-known solutions of two differential equations. Let N be the set of all natural numbers and let $n \in N$. Further, $j = \sqrt{-1}$ and $R = R(r)$ is twice differentiable and continuous function in the domain of definition.

The differential equation of the second order (model #1)

$$r^2 \frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} - n^2 R = 0 \tag{1}$$

has two linearly independent solutions

$$R_1(r) = r^n \quad \text{and} \quad R_2(r) = r^{-n}. \tag{2}$$

Similary, solutions of the differential equation (model # 2)

$$r^2 \frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} + n^2 R = 0 \tag{3}$$

are

$$R_1(r) = r^{lk} \quad \text{and} \quad R_2(r) = r^{-jn} \tag{4}$$

and can be rewritten in the following forms

$$R_1(r) = \cos(n \ln r) \quad \text{and} \quad R_2(r) = \sin(n \ln r). \tag{5}$$

Introduction

Electric scalar potential within the infinitely long tubes, coaxial tubes or groves with the cross sections that contain circles or parts of circles, can be obtained by integration of Laplace's equation in cylindrical coordinates r, θ, z. In the problems under consideration, electric scalar potential $\varphi(r,\theta)$ does not depend on the z coordinate and the two-dimensional Laplace's equation has the form

$$r^2 \frac{\partial^2 \varphi}{\partial r^2} + r\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial \theta^2} = 0. \tag{6}$$

The common technique for solving Laplace's equation is known as the method of separated variables. This method is based on the assumption that the solution is the product of two functions and that each one is the function of only one variable, i.e. $\varphi(r,\theta) = R(r)T(\theta)$. Thus, Laplace's equation reduces to

$$r^2 \frac{\partial^2 R(r)}{\partial r^2} + r\frac{\partial R(r)}{\partial r} + \frac{R(r)}{T(\theta)}\frac{\partial^2 T(\theta)}{\partial \theta^2} = 0. \tag{7}$$

The next step is to assume that

$$\frac{1}{T}\frac{\partial^2 T(\theta)}{\partial \theta^2} = C, \tag{8}$$

where C is an arbitrary constant. Consequently, Laplace's equation will be split into two independent differential equations

$$\frac{\partial^2 T(\theta)}{\partial \theta^2} - CT(\theta) = 0 \quad \text{and} \tag{9}$$

$$r^2 \frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + r\frac{\mathrm{d}R}{\mathrm{d}r} + CR = 0. \tag{10}$$

The choice of C leads to two different models. However, the final solution of Laplace's equation has to satisfy all boundary conditions. Considering all of the boundary conditions, the constant value C has to be chosen to be negative, $C = -n^2$, or positive, $C = n^2$. In the first case the particular solution for $T(\theta)$ is θ-periodical and leads to the model #1 for $R(r)$, and in the second case the particular solution for $R(r)$ is r-periodical and this is the model #2. Nevertheless, in the solution of some problems the value $n = 0$ has to be taken into account, but in our examples this is not the case. The final solution is a linear combination of an infinite number of particular solutions that satisfies all boundary conditions and that solution is unique.

The basic problem. Let us consider infinitely long coaxial electrodes the cross section of which is a circular annulus. The inner and outer electrodes have radii a and b, and potentials $\varphi = 0$ and $\varphi = U$, respectively. The distribution of electric scalar potential does not depend on angular coordinate θ and the solution of the one-dimensional Laplace's equation is trivial and can take the form

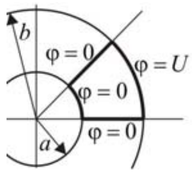
$$\varphi(r) = \gamma \frac{U}{\pi} \ln\frac{r}{a}, \qquad a \le r \le b \tag{11}$$

where

$$\gamma = \frac{\pi}{\ln\frac{b}{a}}. \tag{12}$$

This form of the solution leads to the idea that γ is the constant that is given for any circular annulus and can be used in cases when distribution of electric scalar potential depends on the angular coordinate.

The first problem. The electric scalar potentials of the walls of the circular annulus sector, Figure 1, are prescribed by boundary conditions. The similar problem can be found in [1,2].



Boundary conditions

1) $\varphi = 0$ for $\theta = 0$ and $a \le r \le b$
2) $\varphi = 0$ for $\theta = \alpha$ and $a \le r \le b$
3) $\varphi = 0$ for $r = a$ and $0 \le \theta \le \alpha$
4) $\varphi = U$ for $r = b$ and $0 \le \theta \le \alpha$

Fig 1 Cross section of the circular annulus sector - model #1

It is quite natural to assume that the function $T(\theta)$ is periodical,

(13) $$T_n(\theta) = \sin\frac{n\pi\theta}{\alpha},$$

and such solution ensures that the first and the second boundary conditions are automatically satisfied.

Consequently, this means that we have model #1 for $R(r)$,

(14) $$R_n(r) = r^{\frac{n\pi}{\alpha}}\left(1 - \left(\frac{a}{r}\right)^{\frac{2n\pi}{\alpha}}\right),$$

and, thus, the third boundary condition is satisfied. The final solution of Laplace's equation that satisfies all boundary conditions except the last is a linear combination of an infinite number of particular solutions,

(15) $$\varphi(r,\theta) = \sum_{n=1}^{\infty} C_n R_n(r) T_n(\theta),$$

where the constants C_n are undetermined at present. These constants can be determined by satisfying the last boundary condition expressed in the form of Fourier's series,

(16) $$\varphi(r = b,\theta) = U = \frac{4U}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)\pi\theta/\alpha)}{2n-1},$$

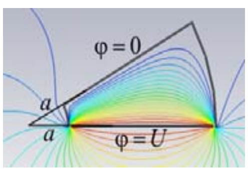
and setting it equal to the final form of the solution for $r = b$. Thus, we have

(17) $$C_{2n} = 0,$$

(18) $$C_{2n-1} = \frac{4U}{(2n-1)\pi}\left(b^{\frac{(2n-1)\pi}{\alpha}} - a^{\frac{2(2n-1)\pi}{\alpha}}\right)^{-1}.$$

Substituting (13), (14), (17) and (18) in (15), we obtain the final solution of Laplace's equation that satisfies all prescribed boundary conditions.

The second problem. Consider the circular annulus sector, Fig. 2, the cross-section of which is the same as in the previous problem. In this case boundary conditions are different from the previous ones and define the electric scalar potential on the walls of the sector as follows.



Boundary conditions

1) $\varphi = U$ for $\theta = 0$ and $a \le r \le b$
2) $\varphi = 0$ for $\theta = \alpha$ and $a \le r \le b$
3) $\varphi = 0$ for $r = a$ and $0 \le \theta \le \alpha$
4) $\varphi = 0$ for $r = b$ and $0 \le \theta \le \alpha$

Fig 2 Cross section of the circular annulus sector - model #2

It is natural that the function $R(r)$ has to be periodical, i.e. we have model #2. Let us introduce the following notations (see Appendix), [3]:

(19) $$\gamma_n = n\gamma = \frac{n\pi}{\ln\frac{b}{a}}$$

(20) $$\mathrm{cl}_n\left(\frac{r}{a}\right) = \cos\left(\gamma_n \ln\frac{r}{a}\right)$$

(21) $$\mathrm{sl}_n\left(\frac{r}{a}\right) = \sin\left(\gamma_n \ln\frac{r}{a}\right)$$

The constant γ arises from the basic problem and ensures that the pair of particular solutions (5) is orthogonal in the domain of definition, $a \le r \le b$, with weighting function $1/r$. The solution (15) of Laplace's equation (6) that satisfies all of the boundary conditions except the first has the form

(22) $$\varphi(r,\theta) = \sum_{n=1}^{\infty} C_n \,\mathrm{sl}_n\left(\frac{r}{a}\right)\frac{\sinh(\gamma_n(\alpha-\theta))}{\sinh(\gamma_n\alpha)}$$

where constants C_n are undetermined at present.
The first boundary condition,

(23) $$f(r) = \varphi(r,0) = \begin{cases} U, & r \in (a,b) \\ 0, & r \notin (a,b) \end{cases}$$

has to be expressed in the form of Fourier's series

(24) $$f(r) = \sum_{n=1}^{\infty} a_n \,\mathrm{sl}_n\left(\frac{r}{a}\right)$$

where constants a_n can be obtained using orthogonal properties (see Appendix)

(25) $$a_n = \frac{2}{\ln(\mathrm{b/a})}\int_{\mathrm{a}}^{\mathrm{b}}\frac{1}{r} f(r)\,\mathrm{sl}_n\left(\frac{r}{a}\right)\mathrm{d}r$$

When we substitute (24) in (25) and use the integration formula (see Appendix) we can compare obtained result with expression (22) for $\theta = 0$. Thus we have

(26) $$C_{2n} = 0 \quad \text{and} \quad C_{2n-1} = \frac{4U}{(2n-1)\pi}.$$

Finally, the distribution of electric scalar potential within the groove is

(27) $$\varphi(r,\theta) = \frac{4U}{\pi}\sum_{n=1}^{\infty}\frac{\mathrm{sl}_{2n-1}(r/a)}{2n-1}\frac{\sinh(\gamma_{2n-1}(\alpha-\theta))}{\sinh(\gamma_{2n-1}\alpha)}$$

It is obvious that boundary conditions $\varphi(r,\alpha) = 0$, $\varphi(a,\theta) = 0$, $\varphi(b,\theta) = 0$ are satisfied. The first boundary condition

(28) $$\varphi(r,0) = \frac{4U}{\pi}\sum_{n=1}^{\infty}\frac{\mathrm{sl}_{2n-1}(r/a)}{2n-1}$$

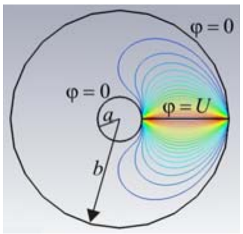
corresponds to the well-known result

(29) $$\varphi(r,0) = \sum_{n=1}^{\infty}\frac{\sin((2n-1)x)}{2n-1} = \frac{\pi}{4} \quad \text{for} \quad 0 < x < \pi;$$

consequently, this means that $a < r < b$ holds. However, the first boundary condition and limits of integration show otherwise. This is not ambiguous because there is always an infinitely small gap between electrodes on different potentials and this remark also applies for the first problem. Also, this difference appears whenever a pulse function has to be expanded into the series.

Application of circular annulus functions

The flat electrode on the potential $\varphi = U$ is placed into the circular annulus. The electrode is perpendicular to the walls that have the potential $\varphi = 0$, Fig 3. Direct application of presented method immediately leads to the final results.



Boundary conditions

$\varphi = 0$ for $r = a$ and $0 \le \theta \le 2\pi$

$\varphi = U$ for $\theta = 0$ and $a \le r \le b$

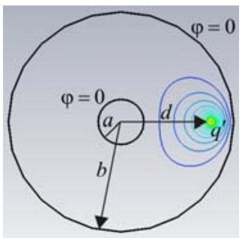
$\varphi = 0$ for $r = b$ and $0 < \theta < 2\pi$

$\dfrac{\partial \varphi}{\partial \theta} = 0$ for $\theta = \pi$

Fig 3 Electrode in circular annulus

(30) $$\varphi(r,\theta) = \frac{4U}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{\cosh\left(\gamma_{2n-1}\left(|\theta| - \pi\right)\right)}{\cosh\left(\gamma_{2n-1}\pi\right)} \mathrm{sl}_{2n-1}\left(\frac{r}{a}\right)$$

As a second example consider line charge q' inside the circular annulus the walls of which have potential $\varphi = 0$, Fig 4.



Boundary conditions

$\varphi = 0$ for $r = a$ and $0 \le \theta \le 2\pi$

$\varphi = 0$ for $r = b$ and $0 \le \theta \le 2\pi$

Fig 4 Line charge in circular annulus

This problem can be solved by multiple use of the theorem of images. In this case we have two concentric perfectly conducting cylindrical mirrors. In the light of this work, Poisson's equation has to be sold,

(31) $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = -\frac{q'}{r\varepsilon}\delta(r-d)\delta(\theta),$$

where δ is Dirac's delta function.

The first step is to solve the corresponding homogenous equation and this is model #2. The next step is to apply Lagrange's method of constants variation. Thus, the time solution is

(32) $$\varphi(r,\theta) = \frac{q'}{\pi\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cosh\left(\gamma_n\left(|\theta| - \pi\right)\right)}{\cosh\left(\gamma_n \pi\right)} \mathrm{sl}_n\left(\frac{d}{a}\right) \mathrm{sl}_n\left(\frac{r}{a}\right)$$

Appendix

Definitions

$k,m,n \in N$, $a,b,r \in R$, $a \le r \le b$

$$r^2 \frac{\mathrm{d}^2 R(r)}{\mathrm{d}r^2} + r\frac{\mathrm{d}R(r)}{\mathrm{d}r} + \gamma_n^2 R(r) = 0$$

where

$$\gamma_n = n\gamma = \frac{n\pi}{\ln\dfrac{b}{a}}.$$

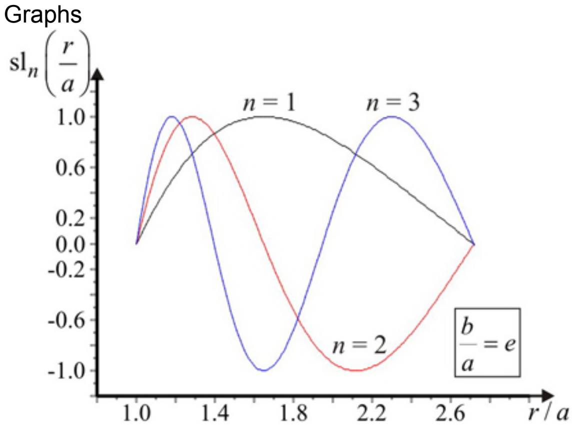
The pair of linearly independent solutions is

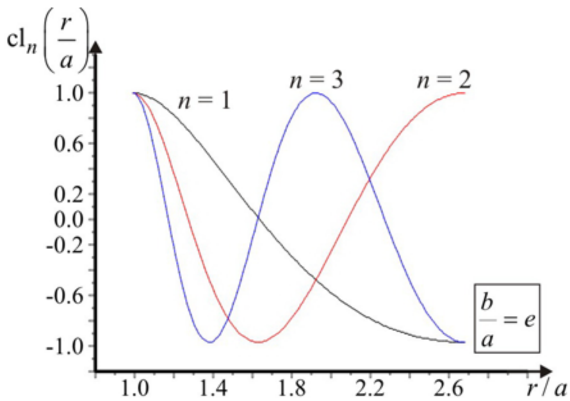
$$\mathrm{cl}_n\left(\frac{r}{a}\right) = \cos\left(\gamma_n \ln\frac{r}{a}\right),$$

$$\mathrm{sl}_n\left(\frac{r}{a}\right) = \sin\left(\gamma_n \ln\frac{r}{a}\right).$$

The most properties of these functions follow directly from those of circular functions by use of the above definitions, [4].

Graphs





Limiting Values

$$\lim_{a \to 0} \mathrm{cl}_n\left(\frac{r}{a}\right) = (-1)^n, \quad \lim_{a \to 0} \mathrm{sl}_n\left(\frac{r}{a}\right) = 0$$

Zeros

$$s_{n,k} = a\left(\frac{b}{a}\right)^{\frac{k-1}{n}}, \quad c_{n,k} = a\left(\frac{b}{a}\right)^{\frac{2k-1}{2n}},$$

$$\frac{s_{n.k+1}}{s_{n,k}} = \frac{c_{n.k+1}}{c_{n,k}} = \left(\frac{b}{a}\right)^{\frac{1}{n}}.$$

Differentiation Formulas

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{sl}_n\left(\frac{r}{a}\right) = \frac{\gamma_n}{r}\mathrm{cl}_n\left(\frac{r}{a}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\mathrm{cl}_n\left(\frac{r}{a}\right) = -\frac{\gamma_n}{r}\mathrm{sl}_n\left(\frac{r}{a}\right)$$

Wronskian

$$W(r)=\frac{\gamma_n}{r}$$

Integration Formulas

$$\int \frac{1}{r}\,\mathrm{cl}_n\left(\frac{r}{a}\right)\ \mathrm{d}r=\frac{1}{\gamma_n}\,\mathrm{sl}_n\left(\frac{r}{a}\right)$$

$$\int \frac{1}{r}\,\mathrm{sl}_n\left(\frac{r}{a}\right)\ \mathrm{d}r=-\frac{1}{\gamma_n}\,\mathrm{cl}_n\left(\frac{r}{a}\right)$$

$$\int_a^b \frac{1}{r}\mathrm{sl}_m\left(\frac{r}{a}\right)\mathrm{cl}_n\left(\frac{r}{a}\right)\mathrm{d}r=\begin{cases}0, & m=n\\ \left(1-(-1)^{m+n}\right)\dfrac{\gamma_m}{\gamma_m^2-\gamma_n^2}, & m\neq n\end{cases}$$

Orthogonal Property

$$\int_a^b \frac{1}{r}\mathrm{sl}_m\left(\frac{r}{a}\right)\mathrm{sl}_n\left(\frac{r}{a}\right)\mathrm{d}r=$$

$$=\int_a^b \frac{1}{r}\,\mathrm{cl}_m\left(\frac{r}{a}\right)\mathrm{cl}_n\left(\frac{r}{a}\right)\mathrm{d}r=\begin{cases}0, & m\neq n\\ \dfrac{n\pi}{2\gamma_n}, & m=n\end{cases}$$

Conclusion

The distribution of electric potential in exterior domain can be obtained by the method of inversion. Without any difficulties, the obtained results can be expanded to an annulus bounded by two confocal ellipses. Likewise, the same method can be applied to waveguides of atypical cross sections.

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