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Eddy currents computation on the basis of variation principle combined with invariant approximations technique

Abstract. *The application of variational principle combined with invariant approximation techniques to eddy currents computation has been discussed. The important advantages of the approach are, firstly, that it allows us to avoid the operation of differentiation of static characteristics of medium what usually results in poor convergence and, secondly, that obtained mathematical model preserves tensor character of initial equations.*

Streszczenie. *W artykule zaproponowano zastosowanie metody wariacyjnej w połączeniu z aproksymacją niezmienniczą do symulacji rozpływu prądów wirowych. Zaletą takiego podejścia jest brak parametrów materiałowych w postaci otwartej w funkcjonale na magnetyczny potencjał wektorowy, co pozwoliło uniknąć operacji ich różniczkowania w procesie przekształcania otrzymanych równań całkowych w układ równań algebraicznych.* ***(Obliczanie prądów wirowych w oparciu o metodę łączącą podejście wariacyjne z aproksymacją niezmienniczą)***

Keywords: variation, invariant approximation, eddy current, static and differential parameter
Słowa kluczowe: metoda wariacyjna, aproksymacja niezmiennicza, prądy wirowe, parametry statyczne i różnicowe

Introduction

The operation of many electromagnetic devices depends on the circulation of eddy currents in their conducting parts and this is why the numerical solution of this problem has become an important research area during last decades [1, 2]. Eddy currents arise in conductive parts of electrical apparatus penetrated by alternating magnetic flux. As many other things in the world they can either cause damage (e.g. overheating, power losses) or be used in measure and inspection devices. The detection and measurements of the strength of the magnetic fields produced by the eddy currents makes it possible for us to learn things about conductive materials without even contacting them (e.g. to measure their thickness and conductivity). Eddy current is used in the nondestructive testing of ferromagnetic and non-ferromagnetic materials. The sensors based on the theory of eddy currents are very sensitive for the detection and analysis of gradual-type defects such as thinning, erosion, material characteristically changes (absolute probes) as well as of local defects such as corrosion, pitting, vibration damages, cracks (differential probes). The right detection may be crucial for human life safety, e. g. by aircraft engine disk inspection.

In our paper we propose an effective way of eddy current field simulation based on the combination of variation methodology with the technique of invariant approximations.

Application of Variation Principle

The eddy current problem is obtained from Maxwell equations by assuming that the frequency is low enough as to neglect the electric displacement in Ampere's Law – so called quasistatic assumption [3]. In general case, the energetic functional developed [5] for electromagnetic field analysis (which corresponds to weak formulation of appropriate Maxwell's equations) in a domain of the volume V filled by anisotropic nonlinear non-homogenous hysteresis medium can be written in the form:

$$(1)\qquad W = \int_V dV \int_{\bar{B}_0}^{\bar{B}} \bar{H} d\bar{B} - \int_V dV \int_{\bar{A}_0}^{\bar{A}} (\partial \bar{D}/\partial t + \bar{J}) \cdot \overline{dA} + \int_V dV \int_{grad\varphi_0}^{grad\varphi} (\bar{D} + \int_{t_0}^{t} \bar{J} dt) \cdot d(grad\varphi) + \int_V \varphi \rho_0 dV + C,$$

where: C - a constant that depends only on initial and boundary conditions.

For the eddy current problem the following assumptions are in effect:

$$(2)\qquad \partial \bar{D}/\partial t = 0\,; \qquad \varphi = 0\,.$$

The second assumption was firstly used in [4]; in such case we are dealing with modified magnetic vector potential. In this particular case the form of the energetic functional is

$$(3)\qquad W_c = \int_V dV \int_{\bar{B}_0}^{\bar{B}} \bar{H} d\bar{B} - \int_V dV \int_{\bar{A}_0}^{\bar{A}} \bar{J} d\bar{A} + C$$

Taking into account that electric field intensity $\bar{E} = -\partial \bar{A}/\partial t$, we can see that the first part of the functional (3) represents the energy of magnetic field and the second one represents the Joule's losses. Thus, the requirement of minimization of this fuctional substantiated in [5] is consistent with the principle of least action. Therefore, solution of the eddy current problem coincides with the minimum of the functional

$$(4)\qquad W_c' = \int_V w \cdot dV + \int_V \chi \cdot dV\,,$$

where: w - magnetic energy density; χ - Joule's losses density.

Applying formulae of vector analysis one obtains the variation of the functional (4) for magnetic vector potential variation $\delta\bar{A}$ in the following form

$$(5)\quad \delta W_c' = \int_V (dw/d\bar{B} \cdot \delta(rot\bar{A}))dV + \int_V (d\chi/d\bar{A} \cdot \delta A)dV =$$
$$= \int_V \bar{H} \cdot \delta(rot\bar{A})dV - \int_V \bar{J} \cdot \delta\bar{A} dV = \int_V div(\delta\bar{A} \times \bar{H}) +$$
$$\int_V rot\bar{H}\delta\bar{A} \cdot dV - \int_V \bar{J}\delta\bar{A} \cdot dV = \oint_S (\bar{H} \times \delta\bar{A})\overline{dS}$$
$$+ \int_V rot\bar{H}\delta\bar{A} \cdot dV - - \int_V \bar{J}\delta\bar{A} \cdot dV = \int_V (rot\bar{H} - \bar{J})\delta\bar{A} dV\,,$$

since $\delta\bar{A}$ =0 on the boundary S of the domain.

The minimum is found under the condition $\delta W_c' = 0$, i.e. $rot\bar{H} = \bar{J}$, because $\delta\bar{A}$ is arbitrary in all points of the domain. Hence the minimization of the energetic functional satisfies Ampere's law for the eddy current problem. The method used for the minimization is finite elements method.

Application of invariant approximations technique to construction of finite elements

The fundamental idea of the finite elements method [7] is to subdivide the domain to be studied into small subregions called finite elements (FE). In such way the energetic functional (4) is represented by the sum of separate integrals taken over every FE:

$$(6)\qquad W_c' = \sum_{m=1}^{M} \int_{V_m} (w + \chi) dV\,,$$

where: V_m - the volume of m-th finite element.

Unknown scalar or vector functions to be found are approximated in each finite element by simple functions called shape functions. A shape function is a continuous function defined over a single finite element. The shape functions of individual finite elements are combined into global shape functions, also called basis functions. There are nodal and edge elements that are utilized depending on the physical properties of a task. The technique of invariant approximations developed by Ukrainian scientist R. Filc [8] applied to finite elements method allows us to construct finite elements whose shape functions are invariant with respect to linear transformations of local and global coordinate frame. This technique states that the tensor character of Maxwell's equations cannot be lost on the stage of their replacement by corresponding discrete analogues. Let us explain this requirement using a bilinear Lagrange rectangle shown in Fig. 1.

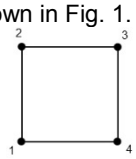

Fig. 1. Bilinear Lagrange rectangular finite element.

Within this element a scalar function U is approximated as

(7) $$U = \sum_{p=1}^{4} N_p U_p ,$$

where N_p is the nodal shape function corresponding to node *p*. When we substitute the formulae of the corresponding shape functions, we receive a general expression of the sought function in the form

(8) $$U = u_1 + u_2 x + u_3 y + u_4 xy .$$

Let us assume that the nodal values of the scalar function are $U_1 = 0$; $U_2 = 1$; $U_3 = 5$; $U_4 = 2$ and find values of the function U in the points 5 and 6 using two different coordinate frames shown in Fig. 2 and Fig. 3 and connected with each other by linear transformations (displacement and rotation).

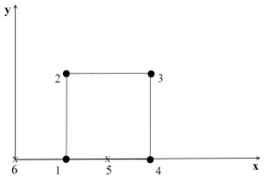

Fig. 2. The first coordinate frame used for approximation of function *U* in the points 5 and 6.

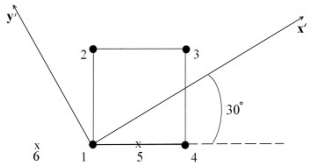

Fig. 3. The second coordinate frame used for approximation of function *U* in the points 5 and 6.

The approximation of the sought function U in different coordinate frames gives following results:

1-st coordinate frame	2-nd coordinate frame
$U_5 = 1$	$U_5 = 1 + \sqrt{3}/8$
$U_6 = -2$	$U_6 = -2 - \sqrt{3}$

For the point 5 the difference between the aproximated values calculated in two different coordinate frames is of 22%; for the point 6 the difference is of 87%. One can see that approximation results depend on the chosen coordinate frame what is principally unacceptable because one of the most important requerements of scientific research is its objectivity.

The same dependence of approximation results on a chosen coordinate frame has been revealed for hexahedral finite elements (Fig. 4).

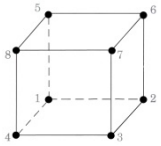

Fig. 4. Hexahedral finite element.

That is why we have constructed finite elements and obtained the mathematical model of eddy-current problem in accordance with invariant approximation technique [6]. Its application allowed us to construct finite elements that

a) are invariant with respect to linear transformation of a coordinate frame;

b) automatically satisfy the boundary conditions on the edge of the part made of conducting material – so called surface FE.

Mathematical model of eddy-current problem

After the domain's subdivision into *M* invariant finite elements the energetic functional (6) can be written in the form:

(9) $$W_c' = \sum_{m=1}^{M} \sum_{p=1}^{P} q_{mp}(w_{mp} + \chi_{mp}) = \sum_{m-1}^{M} F_m ,$$

where: P - freedom degree of the m-th FE; q_{mp} - coefficients that depend only on the geometry of the *m*-th FE; F_m - the contribution of the *m*-th FE into the entire energetic functional.

In accordance with invariant approximations technique [6] the distribution of any unknown function and its differential operator within *m*-th finite element is represented by Taylor's vector of appropriate degree that consists of *P* elements.

(10); $$A_i[x,y,z] = \vec{T} T_m^{-1} \vec{A}_{im*} = \vec{K}[x,y,z]\vec{A}_{im*} \qquad i = x,y,z$$

(11) $$rot\overline{A}[x,y,z] = \vec{T}\overline{N}T_m^{-1} \times \vec{\vec{A}}_{m*} = \vec{\vec{R}}[x,y,z] \times \vec{\vec{A}}_{m*},;$$

where: $\overline{A} = \bar{i}A_x + \bar{j}A_y + \bar{k}A_z$ - vector magnetic potential in any point within the *m*-th FE; $\vec{T}$ - Taylor's vector for any point within the *m*-th FE; T_m^{-1} - inverse Taylor's matrix of the *m*-th FE; $\vec{\vec{A}}_m = \bar{i}\vec{A}_{xm} + \bar{j}\vec{A}_{ym} + \bar{k}\vec{A}_{zm}$ - nodal vectors of vector magnetic potential for the *m*-th FE; $\overline{N}$ - Hamilton's matrix; $\vec{\vec{R}}[x,y,z] = \bar{i}\vec{R}_x + \bar{j}\vec{R}_y + \bar{k}\vec{R}_z$ - difference analog of Hamilton's operator for any the *m*-th FE; the "$*$" symbol means transponation, thus $\vec{T}$, $\vec{\vec{A}}_m$, $\vec{\vec{R}}$ are row-representations of appropriate vectos and $\vec{T}_*$, $\vec{\vec{A}}_{m*}$, $\vec{\vec{R}}_*$ are column-representations of the same vectors.

Since the proposed algorithm requires two consequent differentiation operations we also use other representation of sought functions:

(12) $$A_i[x,y,z] = \vec{A}_{im}\vec{K}_*[x,y,z] \qquad i = x,y,z$$

(13) $$rot\overline{A}[x,y,z] = \vec{\vec{A}}_m \times \vec{\vec{R}}_*[x,y,z] .$$

Taking into account (11), (13) one obtains formulae for components of magnetic flux density in any point within the *m*-th finite element:

(14) $B_x[x,y,z] = -\vec{R}_z\vec{A}_{ym*} + \vec{R}_y\vec{A}_{z*} = -\vec{A}_{ym}\vec{R}_{zm*} + \vec{A}_{zm}\vec{R}_{y*}$;

(15) $B_y[x,y,z] = -\vec{R}_x\vec{A}_{zm*} + \vec{R}_z\vec{A}_{xm*} = -\vec{A}_{zm}\vec{R}_{x*} + \vec{A}_{xm}\vec{R}_{z*}$;

(16) $B_z[x,y,z] = -\vec{R}_y\vec{A}_{xm*} + \vec{R}_x\vec{A}_{ym*} = -\vec{A}_{xm}\vec{R}_{y*} + \vec{A}_{ym}\vec{R}_{x*}$.

To find the minimum of the functional (9) we have to solve the following set of equations:

(17) $\partial W_c'/\partial A_x = 0$; $\partial W_c'/\partial A_y = 0$; $\partial W_c'/\partial A_z = 0$.

The discrete form of the system (17) is as follows:

(18) $\sum_{m=1}^{M}\vec{\Phi}_{mxm*} = 0$; $\sum_{m=1}^{M}\vec{\Phi}_{mym*} = 0$; $\sum_{m=1}^{M}\vec{\Phi}_{mzm*} = 0$,

where: $\vec{\Phi}_{mim*} = \partial F_m/\partial\vec{A}_{im}$ $(i = x,y,z)$.

Taking (3) and (9) into account, let us find the derivative of the contribution F_m of the *m*-th FE with respect to its nodal potentials in the form of column-vectors:

$$(19)\quad \partial F_m/\partial\vec{A}_{xm} = \sum_{p=1}^{P} q_{mp}\partial/\partial\vec{A}_{xm}\Big(\int_{\vec{B}_0}^{\vec{B}_{mp}}\vec{H}d\vec{B} - \int_{\vec{A}_0}^{\vec{A}_{mp}}\vec{J}d\vec{A}\Big) =$$

$$= \sum_{p=1}^{P} q_{mp}\Big(H_{xmp}\frac{\partial B_{xmp}}{\partial\vec{A}_{xm}} + H_{ymp}\frac{\partial B_{ymp}}{\partial\vec{A}_{xm}} + H_{zmp}\frac{\partial B_{zmp}}{\partial\vec{A}_{xm}} -$$

$$- J_{xmp}\frac{\partial A_{xmp}}{\partial\vec{A}_{xm}} - J_{ymp}\frac{\partial A_{ymp}}{\partial\vec{A}_{xm}} - J_{zmp}\frac{\partial A_{zmp}}{\partial\vec{A}_{xm}}\Big) =$$

$$= \sum_{p=1}^{P} q_{mp}(H_{ymp}\vec{R}_{z*} - H_{zmp}\vec{R}_{y*} - J_{xmp}\vec{K}_{mp*}).$$

Analogically

(20) $\partial F_m/\partial\vec{A}_{ym} = \sum_{p=1}^{P} q_{mp}(H_{zmp}\vec{R}_{x*} - H_{xmp}\vec{R}_{z*} - J_{ymp}\vec{K}_{mp*})$;

(21) $\partial F_m/\partial\vec{A}_{zm} = \sum_{p=1}^{P} q_{mp}(H_{xmp}\vec{R}_{y*} - H_{ymp}\vec{R}_{x*} - J_{zmp}\vec{K}_{mp*})$;

Since the node under global number *k* may correspond to different *p* numbers in adjacent FE-s, the full derivative comprises contributions from different FE-s.

The obtained nonlinear system is solved by means of Newton's method that requires us to calculate Jacobi matrix during each iteration stage. To compute the contributions of every FE to the Jacobi matrix we need to perform the differentiation of the column-vectors $\vec{\Phi}_{mxm*}$, $\vec{\Phi}_{mym*}$, $\vec{\Phi}_{mzm*}$ with respect to column-vectors $\vec{A}_{xm*}$, $\vec{A}_{ym*}$, $\vec{A}_{zm*}$:

$$\frac{\partial\vec{\Phi}_{mxm*}}{\partial\vec{A}_{xm*}} = \sum_{p=1}^{P} q_{mp}\Big(\vec{R}_{z*}\frac{\partial H_{xmp}}{\partial\vec{A}_{xm*}} - R_{y*}\frac{\partial H_{zmp}}{\partial\vec{A}_{xm*}} - \vec{K}_{mp*}\frac{\partial J_{xmp}}{\partial\vec{A}_{xm*}}\Big)$$

$$(22)\quad = \sum_{p=1}^{P} q_{mp}\Big(\vec{R}_{z*}\frac{\partial H_{ymp}}{\partial B_{ymp}}\frac{\partial B_{ymp}}{\partial\vec{A}_{xm*}} + \vec{R}_{z*}\frac{\partial H_{ymp}}{\partial B_{zmp}}\frac{\partial B_{zmp}}{\partial\vec{A}_{xm*}} -$$

$$- R_{y*}\frac{\partial H_{zmp}}{\partial B_{ymp}}\frac{\partial B_{ymp}}{\partial\vec{A}_{xm*}} - R_{y*}\frac{\partial H_{zmp}}{\partial B_{zmp}}\frac{\partial B_{zmp}}{\partial\vec{A}_{xm*}} - \vec{K}_{mp*}\frac{\partial J_{xmp}}{\partial E_{xmp}}\frac{\partial E_{xmp}}{\partial\vec{A}_{xm*}}\Big)$$

Other derivatives $\partial\vec{\Phi}_{mim*}/\partial\vec{A}_{im*}$ $(i = y,z)$ as well as mutual derivatives $\partial\vec{\Phi}_{mim*}/\partial\vec{A}_{jm*}$ $(i = x,y,z,\ j = x,y,z,\ i \neq j)$ are to be received in the same way but their formulae are too length. To find these derivatives we need to have the vector dependencies $\vec{H} = \vec{H}[\vec{B}]$, $\vec{J} = \vec{J}[\vec{E}]$ that split into such scalar sets

(23) $H_i = H_i[B_x,B_y,B_z]$, $J_i = J_i[E_x,E_y,E_z]$ $(i = x,y,z)$.

The derivative of vector magnetic potential with respect to time is represented by back-differentiation expression:

(24) $\partial\vec{A}/\partial t = a_0\vec{A} + \vec{C}_A$,

where: a_0 - a coefficient that depends on the order of the used back-differentiation formula; $\vec{C}_A$ - a constant for a given iteration stage that depends on the values of the potential on the previous stages and on the order of the used back-differentiation formula.

Utilizing (14)-(16), (23), (24) we obtain final formulae for contributions of the *m*-th FE to Jacobi matrix:

$$(25)\quad \partial\vec{\Phi}_{mim*}/\partial\vec{A}_{im*} = \sum_{p=1}^{P} q_{mp}(\vec{R}_{k*}\nu_{jjmp}\vec{R}_k - \vec{R}_{k*}\nu_{jkmp}\vec{R}_j -$$

$$- R_{j*}\nu_{kjmp}\vec{R}_k + \vec{R}_{j*}\nu_{kkmp}\vec{R}_j + \vec{K}_{mp*}a_0\gamma_{iimp}\vec{K})$$;

$$(25)\quad \partial\vec{\Phi}_{mim*}/\partial\vec{A}_{jm*} = \sum_{p=1}^{P} q_{mp}(-\vec{R}_{k*}\nu_{jimp}\vec{R}_k + \vec{R}_{k*}\nu_{jkmp}\vec{R}_i +$$

$$+ R_{j*}\nu_{kimp}\vec{R}_k - \vec{R}_{j*}\nu_{kkmp}\vec{R}_i + \vec{K}_{mp*}a_0\gamma_{ijmp}\vec{K})$$;

where: ν_{iimp}, ν_{ijmp} are self and mutual specific magnetic reluctance in the *p*-th node of the *m*-th FE; γ_{iimp}, γ_{ijmp} are self and mutual conductivity in the *p*-th node of the *m*-th FE; i,j,k take in turn the meanings x,y,z and cannot be equal to each other.

Features of the proposed approach

The main difference between the proposed variation approach and other known methodologies consists in the fact that our formulation does not use medium characteristics (permeability, conductivity) in explicit form what allows us to avoid the operation of their differentiation while solving the algebraic set of equations received in the process of numerical modelling. The minimization of the obtained functional has been conducted with application of differential parameters of the medium.

The second advantage of our approach consists in the application of technique of invariant approximations that guarantees the preservation of tensor character of Maxwell's equations while constructing their numerical counterpart. This technique allows us to construct finite elements of any order of discretization order whose nodal shape functions are invariant with respect to linear transformations of local (and global) coordinate frames and which satisfy boundary conditions automatically.

REFERENCES

[1] Krawczyk A., Tegopoulos J. A. Numerical modelling of eddy currents, *Monographs in Electrical & Electronic Engineering,* 1993 (32), Oxford Science Publications, 116 p.

[2] Biro O., Preis, K. An efficient time domain method for nonlinear periodic eddy current problems, *IEEE Transactions on Magnetics*, 2006 (42), No 4, 695–698.

[3] Bermudez, A., Bullon, J., Pena, F., Salgado, P. A numerical method for transient simulation of metallurgical compound electrodes. *Finite. Elem. Anal. Des*., 2003 (39), 283-299.

[4[Biro O., Preis K. On the use of the magnetic vector potential in the finite element analysis of three-dimensional eddy currents. *IEEE Transactions on Magnetics,* 1989 (25), No 4, 3145-3159.

[5] Howykowycz M., Filc R. Theoretical principles of variation approach to electromagnetic field analysis. Part 1, *Theoretical Electrical Engineering,* 2007 (59), 87-96.

[6] Howykowycz M., Horyachko V. Application of invariant approximations to variation problem of stationary electric field, *Przegląd Elektrotechniczny,* 2010 (1), 95-97.

[7] Morton K. Lecture Notes on Finite Element Methods. Oxford University Computing Laboratory, 1991.

[8] Filc R. Discrete analogy of the Hamilton's operator. *Mathematical Methods and Physical-Mechanics Fields,* 1986 (24), 20-25.

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