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Trust Region Method for Equilibrium Network Design Problem

***Abstract**. This paper addresses a mathematical program with equilibrium constraints (MPEC) for network design problem with respect to link capacity expansions and signal settings, where stochastic user equilibrium constraints are expressed as variational inequality problem. The gradient of object function is received by sensitivity analysis of parametric variational inequality and a trust region method is presented for MPEC. Finally, numerical calculations are conducted and promising results have shown potential of the proposed method in solving network design problem.*

***Streszczenie.** W artykule opisano algorytm optymalizacji MPEC do tworzenia sieci, biorący pod uwagę możliwą ilość linków i ustawienia sygnału. W celu uwzględnienia równoważności użytkowników stochastycznych, wykorzystano zagadnienie nierówności wariacyjnej. Poprzez analizy nierówności otrzymano gradient funkcji obiektu oraz obszar ufności dla metody MPEC. Przedstawione wyniki obliczeń wykazują obiecujące efekty działania proponowanej optymalizacji. (**Metoda obszaru ufności w projektowaniu sieci równoważnej**).*

Keywords: network design problem, stochastic user equilibrium, trust region method.
Słowa kluczowe: projektowanie sieci, równoważność użytkownika stochastycznego, metoda obszaru ufności.

Introduction

Network design problem (NDP) is to determine the set of link capacity expansions where users' route choice is taken into account. which is one of the most intensive problems in transportation literature. In past decades, a lot of rich work has been done from the associated references. Yang et al1-2 characterized the optimality conditions and derived the corresponding solution methods where the non-smooth approaches have been considered. Lawphongpanich er al3-5 formulate NDP as mathematical program with equilibrium constraints (MPEC), bilevel optimization model respectively. Chiou6-8 presented a series of methods based on subgradient, such as quasi-Newton subgradient projection method, generalized bundle subgradient projection method and conjugate subgradient projection method. Tobin and Patriksson9-13 work on traffic equilibrium by sensitivity analysis.

In this paper, we firstly formulate asymmetric traffic network design problem with signal controlled and capacity constraints based on stochastic user equilibrium (SUE) instead of user equilibrium(UE) as MPEC. The first order sensitivity analysis is conducted and the gradient of variables of interest can be conveniently computed. a trust region method is presented for NDP. Numerical calculations are carried out on a road ollows.

Problem formulation

In this section, NDP is presented with a mathematical program with equilibrium constraints. Firstly, SUE is expressed in terms of variational inequality(VI) where user's route choice is assumed to follow logit assignment principle. Then, first-order sensitivity analysis is conducted for which the gradient of variables of interests is conducted. Finally, an MPEC formulation is presented.

The following Notation will be used.

$G(N,A)$: Directed road network, where N is set of nodes and A is set of links. W: Set of OD pairs. y_a : Link capacity expansion on link a. $y_a^{\min}$, $y_a^{\max}$: Bounds of link capacity expansion on link a. $G_a(y_a)$: Investment cost on link a. θ: Conversion factor from investment cost to travel time. λ_a : Vector of green light as proportions of common cycle time at the exit of link a. $\lambda_a^{\min}, \lambda_a^{\max}$:Vector of bound of green light proportions. f : Vector of link flow. p: Vector of path flow. c: Vector of link flow travel cost. C: Vector of path flow travel cost. s_a : Saturation flow on link a. γ_a : Saturation degree on link a. q: Vector of travel demand. Δ: Link-path incidence matrix. Γ: OD-path incidence matrix. For traffic assignment problem of stochastic user equilibrium, a variational inequality model can be expressed as follows:

(1) $$[\ln p_k^w + \alpha C_k^w(\sum_w \sum_k p_k^w \delta_{ak}^w)](\overline{p_k^w} - p_k^w) \geq 0, \quad \forall \overline{p_k^w} \in D = \left\{ p_k^w \middle| \sum_k p_k^w = q_w, p_k^w > 0 \right\}$$

α is a positive dispersion parameter, which reflects an aggregate measure of drivers' perception of travel costs.

Theorem: Assume link flow travel cost function $C(p)$ is continuous, differentiable and monotone, the solution of (1) follows logit assignment principle.

Proof: The KKT conditions of (1) are:

(2) $$\ln p_k^w + \alpha C_k^w + u_w - v_k^w = 0$$

(3) $$\sum_k p_k^w = q_w$$

(4) $$p_k^w > 0, \quad v_k^w \geq 0, \quad p_k^w v_k^w = 0$$

From (4), $v_k^w = 0$ .With (2), $p_k^w = e^{-\alpha C_k^w - u_w}.q_w = \sum_k p_k^w = e^{-u_w} \sum_k e^{-\alpha C_k^w}$, $\frac{p_k^w}{q_w} = \frac{e^{-\alpha C_k^w}}{\sum_k e^{-\alpha C_k^w}}$, $p_k^w = q_w \frac{e^{-\alpha C_k^w}}{\sum_k e^{-\alpha C_k^w}}$.

This means the solution of (1) follows logit assignment principle. In term of (1), traffic assignment problem with signal settings and link capacity expansions can be concluded as a parametric variational inequality:

(5) $$[\ln p_k^w(y,\lambda) + \alpha C_k^w(p(y,\lambda),y,\lambda)][\overline{p_k^w}(y,\lambda) - p_k^w(y,\lambda)] \geq 0, \forall \overline{p_k^w}(y,\lambda) \in K(y,\lambda) = \left\{ p_k^w(y,\lambda) \middle| \sum_k p_k^w(y,\lambda) = q_w, p_k^w(y,\lambda) > 0 \right\}$$, where

$$C_k^w = \sum_a \delta_{ak}^w c_a(f(y,\lambda),y,\lambda), \quad f_a(y,\lambda) = \sum_w \sum_k \delta_{ak}^w p_k^w(y,\lambda).$$

The KKT conditions of (5) are:

(6)
$$\ln p_k^w(y,\lambda)+\alpha C_k^w(p(y,\lambda),y,\lambda)+u_w-v_k^w=0$$
$$\sum_k p_k^w(y,\lambda)=q_w,$$
$$p_k^w(y,\lambda)>0,\quad v_k^w\ge 0,\quad p_k^w(y,\lambda)v_k^w=0$$

Simplify (6),

(7)
$$\ln p_k^w(y,\lambda)+\alpha C_k^w(p(y,\lambda),y,\lambda)+u_w=0$$
$$\sum_k p_k^w(y,\lambda)=q_w,$$

Introduce

(8)
$$H(p,u,y,\lambda)=\begin{pmatrix}\ln p(y,\lambda)+\alpha\Delta^T c(\Delta p(y,\lambda),y,\lambda)+\Gamma^T u\\ \Gamma p(y,\lambda)-q\end{pmatrix}$$

Denoted (p,u) as z and (y,λ) as β. Therefore the first order sensitivity analysis of equations (8) for β can be derived by

(9)
$$\nabla_z H\nabla_\beta z+\nabla_\beta H=0$$

Where $\nabla_z H=\begin{pmatrix}P^{-1}+\alpha\Delta^T\nabla_z c(\Delta p,\beta)\Delta & \Gamma^T\\ \Gamma & 0\end{pmatrix}=\begin{pmatrix}A & \Gamma^T\\ \Gamma & 0\end{pmatrix}$,

$\nabla_\beta H=\begin{pmatrix}\alpha\Delta^T\nabla_\beta c(\Delta p,\beta)\\ 0\end{pmatrix}$, $P^{-1}=diag(\cdots,\frac{1}{p_k^w},\cdots)$.

From (9), $\nabla_\beta z=-\begin{pmatrix}A & \Gamma^T\\ \Gamma & 0\end{pmatrix}^{-1}\begin{pmatrix}\alpha\Delta^T\nabla_\beta C(\Delta p,\beta)\\ 0\end{pmatrix}$

$=\begin{pmatrix}A^{-1}(I-\Gamma^T(\Gamma A^{-1}\Gamma^T)^{-1}\Gamma A^{-1}) & A^{-1}\Gamma^T(\Gamma A\Gamma^T)^{-1}\\ (\Gamma A^{-1}\Gamma^T)^{-1}\Gamma A^{-1} & -(\Gamma A^{-1}\Gamma^T)^{-1}\end{pmatrix}\begin{pmatrix}\alpha\Delta^T\nabla_\beta c(\Delta p,\beta)\\ 0\end{pmatrix}$. This means

$$\nabla_\beta p=-\alpha A^{-1}(I-\Gamma^T(\Gamma A^{-1}\Gamma^T)^{-1}\Gamma A^{-1})\Delta^T\nabla_\beta c(\Delta p,\beta)$$
$$\nabla_\beta u=-\alpha(\Gamma A^{-1}\Gamma^T)^{-1}\Gamma A^{-1}\Delta^T\nabla_\beta c(\Delta p,\beta).$$

$f=\Delta p$, so

(10) $$\nabla_\beta f=\alpha\Delta A^{-1}(I-\Gamma^T(\Gamma A^{-1}\Gamma^T)^{-1}\Gamma A^{-1})\Delta^T\nabla_\beta c(\Delta p,\beta)$$

An optimization model for NDP can be formulated as

(11) $$\min Z=\sum_a c_a(f(y,\lambda),y,\lambda)f_a(y,\lambda)+\theta\sum_a G_a(y_a)$$

(12) $$s.t\ \ y_a^{\min}\le y_a\le y_a^{\max},\quad\forall a$$

(13) $$\lambda_a^{\min}\le\lambda_a\le\lambda_a^{\max},\forall a$$

(14) $$B_a\lambda_a=b_a,\forall a$$

(15) $$f_a(y,\lambda)\in S(y,\lambda)$$

Where $S(y,\lambda)$ is the solution set of (5).

In constrains (12)~(14), let B and b be the coefficient matrix and corresponding constant vector associated, thus (12)~(14) can be rewritten as the following form : $D\beta=b$, $\beta^{\min}\le\beta\le\beta^{\max}$. Following the results in sensitivity analysis, the first-order partial derivatives can be obtained by (10). Now the model (11)-(15) can be re-expressed as a single-level problem:

(16)
$$\min_\beta\quad Z=Z(\beta)$$
$$s.t.\quad D\beta=b$$
$$\beta^{\min}\le\beta\le\beta^{\max}$$

For (16), the objective function $Z(\beta)$ has no specific form. However, the gradient can be derived by sensitivity analysis.

(17) $$\nabla Z(\beta^k)=\nabla_\beta Z_0(f^k,\beta^k)+\nabla_f Z_0(f^k,\beta^k)\nabla_\beta f$$

Trust region method for NDP

Due to the sensitivity analysis, a trust region method for simultaneously solving signal settings and capacity expansions in (16) can be established. Let B_k be the approximation of Hesse matrix, a quadratic program can be concluded by using quadratic approximation of the objective function $Z(\bullet)$.

(18)
$$\min\quad G=\nabla Z(\beta^k)^T d+\frac{1}{2}d^T B_k d$$
$$s.t.\quad \|d\|_\infty\le\Delta_k$$
$$Ad=0$$
$$\beta^{\min}-\beta^k\le d\le\beta^{\max}-\beta^k$$

where $B_{k+1}=B_k-\frac{B_k u_k u_k^T B_k}{u_k^T B_k u_k}+\frac{v_k v_k^T}{v_k^T v_k}$, $u_k=\beta^k-\beta^{k-1}$, $v_k=\nabla Z(\beta^k)-\nabla Z(\beta^{k-1})$. Let $\Delta_z=Z(\beta^k)-Z(\beta^{k+1})$, $\Delta_G=Z(\beta^k)-G(d^k)$, then

$$\Delta_{k+1}=\begin{cases}\frac{1}{2}\Delta_k, & if\quad \Delta_z<0.1\Delta_G;\\ \Delta_k, & if\quad 0.1\Delta_G\le\Delta_z\le 0.75\Delta_G;\\ 2\Delta_k, & if\quad \Delta_z>0.75\Delta_G.\end{cases}$$

Conclude the above analysis, a new trust region scheme for MPEC is established in the following steps.

Step1 Set initial parameters $\beta^k,\Delta_k,k=1.$

Step2 Solve (5) and let p^k be the solution.

Step3 Compute $\nabla_\beta f$ in (10) $\nabla Z(\beta^k)$ in (17).

Step4 If $\nabla Z(\beta^k)=0$, then stop; otherwise continue.

Step5 Solve (18) and suppose d^k is the solution. Find new iterate $\beta^{k+1}=\beta^k+d^k$. Let $k=k+1$, then go to Step2.

Numerical calculations

In this section, numerical computations are conducted by trust region method in signal-controlled network where example network is shown in Fig.1. In this traffic network, the capacity of links 1,2,3,4 need adjustment and the green light proportions of intersections 4,5,6 need to be assigned.

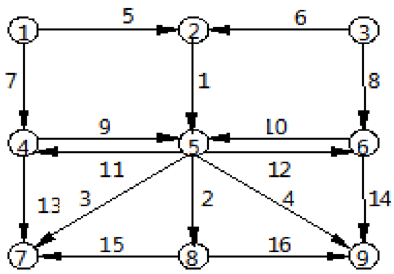

Fig.1.Testing network

The link travel time c_a^0 and link capacity s_a are shown in Table1. Computational results are concluded in Table2 and Table 3.

Table 1 Initial value of c_a^0 and s_a

a	1	2	3	4	5	6	7	8
c_a^0	2	2	3	3	1	1	2	2
s_a	45	45	0	0	35	30	30	35
a	9	10	11	12	13	14	15	16
c_a^0	2	1	2	1	2	1	2	2
s_a	36	40	35	35	30	35	40	40

Table 2 Computational results for $\lambda_a^{\min}$ =0.3, $\lambda_a^{\max}$ =0.7, $y_a^{\min}$ =0, $y_a^{\max}$ =6.5.

θ	($\lambda_{1,4}$,$\lambda_{1,5}$,$\lambda_{1,6}$)	Invest cost	(y_1,y_2,y_3,y_4)	Travel cost
1	(0.49,0.44,0.43)	151.3	(3.9,5.7,1.9,2.6)	192.4
2	(0.50,0.44,0.45)	134.7	(3.0,4.8,2.2,3.1)	201.5
4	(0.51,0.49,0.40)	110.2	(1.9,2.3,0,0)	233.7
8	(0.48,0.44,0.41)	110.2	(1.0,2.2,0,0)	244.9

Table3 Computational results for $\lambda_a^{\min}$ =0.2, $\lambda_a^{\max}$ =0.8, $y_a^{\min}$ =0, $y_a^{\max}$ =10.

θ	($\lambda_{1,4}$,$\lambda_{1,5}$,$\lambda_{1,6}$)	Invest cost	(y_1,y_2,y_3,y_4)	Travel cost
1	(0.44,0.42,0.47)	213.4	(4.7,4.9,2.4,2.1)	220.6
2	(0.48,0.46,0.43)	152.6	(3.5,4.7,1.0,1.0)	232.2
4	(0.46,0.53,0.39)	139.5	(3.2,3.8,0,0)	276.8
8	(0.44,0.48,0.44)	133.8	(2.5,2.8,0,0)	297.4

As it observed is Tables 2 and 3, trust region method receives promising results and show faster convergence.

Conclusions

This paper presents a new model for NDP based on SUE which is expressed as an MPEC program. A trust region scheme is proposed to effectively search for optimal solution. Numerical experiments are conducted on example network, where good performance shown in solving NDP.

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