Guohua Dong[1], Xiangjing An[1], Dewen Hu[1]

College of Mechatronics and Automation, National University of Defense Technology, Changsha, Hunan, China (1)

Contributions to the Horn-Schunck Optical Flow Equations -Part III: Alternating Iteration Algorithms

***Abstract**. The Horn-Schunck equations are a coupled system of two partial differential equations which aim at finding motion information in a given image sequence. Recent results [1, 2] asserted that this system is well-posed and can not be decoupled under any linear transformations. In this paper, two alternating iterative algorithms are proposed to solve this system. These algorithms have three properties: first, at each single iteration, both algorithms consist of two decoupled, scalar equations of elliptic type, driven by the last approximate solutions; second, the particular form of iterations allows analytical solutions expressed via potential integral, Poisson integral, conformal mapping and Feynman-Kac formula; third, exponential convergence of these algorithms are established under mild conditions and the rates of convergence are given with the help of energy inequalities and Banach fixed point principle for contraction mappings. Limitations of these algorithms are discussed.*

***Streszczenie.** W artykule zaproponowano sposób analizy i rozwiązywania równań Horn'a-Schunck'a. Rozwiązanie polega na zastosowaniu dwóch algorytmów o przemiennej iteracji. W każdym kroku obydwa algorytmy składają się z dwóch niezależnych eliptycznych równań skalarnych, bazujących na ostatnim przybliżonym rozwiązaniu. Otrzymane rozwiązanie może być wyrażone poprzez całkę potencjału, całkę Poisson'a, odwzorowanie wiernokątne, formułę Feynman'a-Kac'a. Przedstawiono i omówiono ograniczenia stosowania proponowanych algorytmów. (**Analiza równań Horn-Schunck'a przepływu optycznego – część III: algorytmy o iteracji przemiennej**).*

Keywords: Optical flow, Horn-Schunck equations, alternating iteration, analytical formulae, exponential convergence, rate of convergence.
Słowa kluczowe: przepływ optyczny, równania Horn-Schunck'a, iteracja przemienna, równanie analityczne, zbieżność wykładnicza, stopień konwergencji.

1. Introduction

Let us recall results in the previous two parts which motivate our current work. In part I [1], the Horn-Schunck system [1-4]

$$\Delta u = \lambda[I_x u + I_y v + I_t]I_x \qquad (1)$$
$$\Delta v = \lambda[I_x u + I_y v + I_t]I_y$$

is introduced, which is the Euler-Lagrange equations of the variational problem.

$$(2)\quad \min_{(u,v)} \iint [\|\nabla u\|^2 + \|\nabla v\|^2] dxdy + \lambda \iint [I_x u + I_y v + I_t]^2 dxdy$$

In the [1], the Horn-Schunck system has been proven to be well-posed: its solution exists and is unique, besides, the classical algorithm of Horn-Schunck corresponds to the gradient descent flow of the original variational problem and the Horn-Schunck PDEs describe the stationary state. Globally exponential stability of the algorithm was also established, which automatically implies convergence and uniqueness, and the stability condition and rate of convergence depend explicitly on smoothness of the image sequence and parameters of the algorithm. Two groups of experiments were conducted for the classical Horn-Schunck algorithm where assertions on both convergence and exponential stability were validated under nine different choices of algorithmic parameters, and the approximate sequence of solutions did approach to the ground truth.

The Horn-Schunck equations are in coupled form, and it has been proven in part II [2] that the equations can *not* be decoupled via linear transformations for *generic* image sequences.

In the current part, two alternating iterative algorithms are proposed, both of which are *decoupled* in each step when the previous approximate solution is known. The decoupled form of these new algorithms enables explicit analytical solutions and facilitates quantitative performance analysis. Stability results are established in L^2 sense.

The organization of the rest of this paper is as following. Two alternating iteration algorithms are introduced in Section 2. Section 3 establishes analytical solutions for one step iteration. Globally exponential convergence and rates of convergence are obtained in Section 4. Conclusions and discussions in Section 5 conclude the paper.

2. Two Alternating Iteration Algorithms

Note that in the Horn-Schunck system, the principal parts (*i.e.* partials of second order) have already been in decoupled form, while the couplings only appear on the right hand sides, thus the system has been nearly decoupled. Alternately, we can view the system in the following "scalar" way: given some initial $v^{(0)}$, the first equation is uniformly elliptic with respect to u, and the coefficient before u on the right hand side, *i.e.* λI^2_x is always nonnegative, so by standard results from theory of scalar elliptic equation [5, 6], there exists a unique solution $u^{(0)}=S_1v^{(0)}$ with regularity higher than that of $v^{(0)}$ (under the same standing assumption as in [1], *i.e.* the boundary value of the optical flow is identically to zero); insert $u^{(0)}$ as a known function on the right hand side of the second equation, and reason in the same way, we obtain a unique solution with suitable smoothness, $v^{(1)}=S_2u^{(0)}=S_2S_1v^{(0)}$ .The process stops only if it has reached the equilibrium, *i.e.* the solution of the original system. In this way a sequence of optical flows, $(u^{(n)},v^{(n)}),n=1,2\ldots$, is generated, and satisfies the following recursive system which is *decoupled*

$$\Delta u^{(n)} - \lambda I_x^2 u^{(n)} = \lambda(I_y v^{(n-1)} + I_t)I_x$$
$$(3)\quad \Delta v^{(n)} - \lambda I_y^2 v^{(n)} = \lambda(I_x u^{(n-1)} + I_t)I_y$$
$$v^{(0)} \to u^{(1)} \to v^{(2)} \to u^{(3)} \to \cdots$$

The particular forms of these equations permit us to write down analytical formulae for solutions. In Section 3, we will give a probabilistic form of the solution operators S_1 and S_2 via Feynman-Kac formula. Stability and error estimate will be subjects of Section 4.

In the above algorithm, the solution sequence is driven only by $v^{(0)}$ (and image data), so u and v play different roles. To recover symmetry between them, a pair of initial $(u^{(0)},v^{(0)})$ is given, and the sequence of optical flows is generated as $(u^{(n+1)},v^{(n+1)})=(S_1v^{(n)},S_2u^{(n)}),n=1,2\ldots$.Equivalently, the sequence of approximate solution satisfies the following *decoupled* system

(4) $$\Delta u^{(n)} - \lambda I_x^2 u^{(n)} = \lambda(I_y v^{(n-1)} + I_t)I_x$$
$$\Delta v^{(n)} - \lambda I_y^2 v^{(n)} = \lambda(I_x u^{(n-1)} + I_t)I_y$$

(5) $$\begin{pmatrix} u^{(n-1)} \\ v^{(n-1)} \end{pmatrix} \to \begin{pmatrix} u^{(n)} \\ v^{(n)} \end{pmatrix} = \begin{pmatrix} S_1 v^{(n-1)} \\ S_2 u^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & S_1 \\ S_2 & 0 \end{pmatrix}^n \begin{pmatrix} u^{(0)} \\ v^{(0)} \end{pmatrix}$$

This will be called Algorithm 1 from now on.

If we look at the algorithm every two iterations, it is easy to see that they take the following recursive, decoupled forms：

(6) $$\begin{pmatrix} u^{(n)} \\ v^{(n)} \end{pmatrix} = \begin{pmatrix} S_1 S_2 u^{(n-2)} \\ S_2 S_1 v^{(n-2)} \end{pmatrix}$$

however, in language of equations, this means

(7) $$(\Delta - \lambda I_y^2)\{\tfrac{1}{I_x I_y}[(\Delta - \lambda I_x^2)u^{(n)} - \lambda I_x I_t]\} = \lambda^2 (I_x u^{(n-2)} + I_t)I_y$$
$$(\Delta - \lambda I_x^2)\{\tfrac{1}{I_x I_y}[(\Delta - \lambda I_y^2)v^{(n)} - \lambda I_y I_t]\} = \lambda^2 (I_y v^{(n-2)} + I_t)I_x$$

This system is of order four, so it is merely used in theoretic analysis, not suitable for purpose of practical computation.

The second way to construct a sequence of approximate solutions is as follows：

$$\Delta u^{(n)} = \lambda[I_x u^{(n-1)} + I_y v^{(n-1)} + I_t]I_x \overset{\Delta}{=} g(u^{(n-1)}, v^{(n-1)})I_x$$

(8) $$\Delta v^{(n)} = \lambda[I_x u^{(n-1)} + I_y v^{(n-1)} + I_t]I_y \overset{\Delta}{=} g(u^{(n-1)}, v^{(n-1)})I_y$$

$$\begin{pmatrix} u^{(n-1)} \\ v^{(n-1)} \end{pmatrix} \to \begin{pmatrix} u^{(n)} \\ v^{(n)} \end{pmatrix} = \begin{pmatrix} P_1(u^{(n-1)}, v^{(n-1)}) \\ P_2(u^{(n-1)}, v^{(n-1)}) \end{pmatrix}.$$

This will be called Algorithm 2 throughout this paper.

This system has the advantages that each equation is a standard Poisson equation and decoupled with another. In Section 3, potential integral, harmonic functions and Poisson integral are exploited to express its analytical solution, while stability analysis and error estimate will be given in Section 4.

The above alternating iteration algorithms remind us of the well-known successive iterative method of E. Picard and the continuation method of H. Poincaré.

3. Analytical Solutions for One Step Iteration

3.1 Analytical Formulae for Algorithm 1: Probabilistic Solutions

The well-known Feynman-Kac formula [7] can be used to find the solution operators in Algorithm 1. Since the two operators are similar, we concentrate only on S_1 :

(9) $$\Delta u^{(n)} - \lambda I_x^2 u^{(n)} = \lambda(I_y v^{(n-1)} + I_t)I_x$$
$$v^{(n-1)} \to u^{(n)} = S_1 v^{(n-1)}$$

Let $X(t)=(x,y)+B(t)$ be a Brown motion on the plane starting from (x,y), and $\tau_{(x,y)}$ being the first time when it hits the image boundary. As in Part Ⅰ [1], we assume that the optical flow on the boundary is identically to zero. The Feynman-Kac formula asserts that the following expression gives solution of the above elliptic equation (similar assertion holds for S_2):

(10) $$u^{(n)}(x,y)=S_1 v^{(n-1)} = -E[\int_0^{\tau_{(x,y)}} \tfrac{1}{2}\lambda(I_y v^{(n-1)} + I_t)I_x\Big|_{X(t)} e^{-\int_0^t \frac{1}{2}\lambda I_x^2(X(s))ds} dt]$$

This solution has the following probabilistic meaning: $1/2(\lambda I^2_x)$ is the “killing rate” of $X(t)$ and the “killing time” ξ is independent of $X(t)$, therefore, $e^{-\int_0^t \frac{1}{2}\lambda I_x^2(X(s))ds}$ is the probability that the process is still “*alive*” before it hits the image boundary, *i.e*. the probability that it has not been killed at time t .

Thus, the solution $u^{(n)}(x,y)$ is the expectation of the trajectory function $(-1/2)\lambda(I_y v_{n-1}+I_t)I_x|_{X(t)}$ conditioned that $X(t)$ is still alive before it hits the image boundary. Another (non-probabilistic) interpretation is as follows: if the solution is viewed as the stationary distribution of temperature, *i.e*. the solution of the heat equation $\frac{\partial u}{\partial T} = \Delta u - \lambda I^2_x u - \lambda(I_y v+I_t)I_x$ as $T\to\infty$, then the negative coefficient $-\lambda I^2_x$ means there is cooling from outside and $-\lambda I^2_x u$ is the rate of decent of the temperature.

The probabilistic solution indicates a Monte Carlo implementation of $u^{(n)}=S_1 v^{(n-1)}$. Since stochastic simulation and statistical average should be carried out for each (x,y), the computational load is extremely heavy. Besides, for final precision to be acceptable, say $O(1/N^2)$ (N being number of pixels of input image), the number of trajectories that should be generated at each round is roughly of order $O(N^4)$, thus, $O(N^5)$ sample trajectories are needed to generate in N rounds in order to get an $O(1/N^2)$ precision. This is a common flaw of methods of Monte Carlo type. Moreover, although a single sample trajectory starting from one point can be used via translation to other starting points, however, survival durations and hitting times are different for distinct starting points; if we want to sample a single long enough trajectory so as to use it for every other points after shift, the trajectory must be impractically long such that waste is doomed to be expected, especially for those starting points which are near the boundary, or those starting points which have high killing rates (although by theory of stochastic processes, we always have $E\tau^{(x,y)}<k<\infty$, $\forall$ (x,y) since the image domain is bounded).

The solution has several advantages: first, its analytical form facilitates theoretic analysis; secondly, both the presentation of the strong convergence factor $e^{-\int_0^t \frac{1}{2}\lambda I_x^2(X(s))ds}$ and the smoothing effect of the expectation operation lower the requirement of regularity on $(I_y v^{(n-1)}+I_t)I_x$ for existence of a solution; thirdly, there is no need to solve equations.

The regularity issue has been discussed in Part Ⅰ [1].

3.2 Analytical Formula for Algorithm 2

For Poisson system in Algorithm 2:

(11) $$\Delta u^{(n)} = \lambda(I_x u^{(n-1)} + I_y v^{(n-1)} + I_t)I_x$$
$$\Delta v^{(n)} = \lambda(I_x u^{(n-1)} + I_y v^{(n-1)} + I_t)I_y$$

the following two-stage treatments are standard [5, 6, 8].

First, by using the *fundamental solution* $G(x,y)=log(1/r)$, where $r=\sqrt{x^2+y^2}$, we can formulate a special solution of the system in form of a *potential integral* [6] (boundary condition is yet to be satisfied, and “*” below denotes convolution):

(12) $$\bar{u}^{(n)} = -\tfrac{1}{2\pi}G*(g(u^{(n-1)},v^{(n-1)})I_x),\ \bar{v}^{(n)} = -\tfrac{1}{2\pi}G*(g(u^{(n-1)},v^{(n-1)})I_y)$$

where the function g has been defined in (8).

Second, suppose $u_1^{(0)}$ and $v_1^{(0)}$ are harmonic functions on the image domain, which have the same boundary value as the potential integrals $\bar{u}^{(n)}$ and $\bar{v}^{(n)}$, respectively, so $u^{(n)}=\bar{u}^{(n)} - u_1^{(n)}$, $v^{(n)}=\bar{v}^{(n)} - v_1^{(n)}$ gives the solution of the original problem. From [6] and [1] we know that taking boundary values of these potential integrals can be done in classical sense.

Thus the problem reduces to that of finding harmonic functions on image domain with given boundary values; it is nothing else but the classical *Dirichlet Problem* [5, 6, 8].

Solution of the Dirichlet problem for *upper half plane* is well-known and gives by a Poisson integral. Therefore, if we can find an invertible mapping from the image domain, a *rectangle*, onto the upper half plane, we are done. Fortunately, this is again a classical problem that has been solved in complex analysis: the desired mapping is given by the so-called *conformal mapping*.

The harmonic function defined in the upper half plane with boundary value $\varphi(s_1)$ is given by the following *Poisson integral* [8]

(13) $$\frac{s_2}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(\xi)}{(s_1-\xi)^2+s_2^2}d\xi,\ s=s_1+is_2$$

Let $\varsigma(z)=\int_0^z\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$,$0<k<1$ be *Jacobi's elliptic integral of first type*, it is known [8] that it maps the upper half plane onto a rectangle, more precisely, it maps the following four points on the real axis, *i.e.* 1, $1/k$, $-1/k$ and 1, to the four corners of the rectangle with complex coordinates $\omega_1/2$, $\omega_1/2+i\omega_2$, $-\omega_1/2+i\omega_2$,and $-\omega_1/2$, respectively. Here $\omega_1/2=\zeta(1)$ is the complete elliptic integral and $\omega_2=\int_0^{1/k}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$ is the complementary complete elliptic integral, and its converse function $sn(z)=\zeta^{-1}(z)$ is Jacobi's elliptic function of first type, which maps the closed rectangle back onto the closed upper half plane conformally. If we redefine the image domain as $[-\omega_1/2, \omega_1/2]\times[0, \omega_2]$ through shift and scaling, then the mapping $s=sn(z)$ does the job.

Therefore, the searched harmonic functions, $u_1^{(n)}$ and $v_1^{(n)}$, which have boundary values as that of $\overline{u}^{(n)}$ and $\overline{v}^{(n)}$ on $\Omega=[-\omega_1/2, \omega_1/2]\times[0, \omega_2]$, respectively, are given by

(14) $$\frac{s_2}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(\varsigma(\xi))}{(s_1-\xi)^2+s_2^2}d\xi$$

where $s=s_1+is_2=sn(z)$, $z=x+iy$, and $\varphi(z)=\overline{u}^{(n)}(x,y)|_{\partial\Omega}$ or $\overline{v}^{(n)}(x,y)|_{\partial\Omega}$.

Finally, the explicit solution of Algorithm 2, or the above Poisson equation (8) or (11) with zero boundary value, is

(15) $$u^{(n)}=\overline{u}^{(n)}-u_1^{(n)},\ v^{(n)}=\overline{v}^{(n)}-v_1^{(n)}$$

Justifications of the procedure:

1) Existence and regularity of potential integrals (12): if $g(u^{(n-1)},v^{(n-1)})I_x$ and $g(u^{(n-1)},v^{(n-1)})I_y$ are bounded and integrable, then these potentials $\overline{u}^{(n)}$ and $\overline{v}^{(n)}$ are well defined, which have uniformly continuous derivatives of first order and almost everywhere derivatives of second order; moreover, if $g(u^{(n-1)},v^{(n-1)})I_x$ and $g(u^{(n-1)},v^{(n-1)})I_y$ are in $L^p(p>1)$, then classical Calderón-Zygmund's theory of singular integral operators guarantees that all derivatives of second order are also in L^p; furthermore, if $g(u^{(n-1)},v^{(n-1)})I_x$ and $g(u^{(n-1)},v^{(n-1)})I_y$ are Hölder continuous with order $\alpha>0$, then these potentials are Hölder continuous with order $2+\alpha$ and satisfy the Poisson system everywhere [6]. By Theorem 3 of Part Ⅰ [1], this is indeed the case;

2) Smoothness of the Poisson integral (13) and its continuity when approaching to the boundary: the Poisson integral defines a harmonic function so it is infinitely differentiable in the interior; the Poisson integral converges to its boundary value non-tangentially once the boundary value is itself continuous; since the boundary functions are indeed smooth according to [1], when approaching from the interior domain to the boundary non-tangentially, the convergence holds at every boundary point [8];

3) The above solution can be used both in theoretical analysis and practical numerical computation.

4. Globally Exponential Convergence and Rate of Convergence

According to the *Lax equivalence theorem* cited in Part Ⅰ [1], every approximate solution sequence yielded by a linear iteration is convergent if and only if the iterative scheme is stable, provided that the iteration is consistent. The consistency of these algorithms is apparent since they come from direct modifications of the original system. Only stability is to be established. Here we adopt a strategy different from that has been used in Part Ⅰ [1, 11]: after energy inequalities have been obtained, the celebrated Banach fixed point theorem for contraction mappings is used to show that the distance between two approximate solutions contracts by a factor less then 1 under iteration so that globally exponential stability is concluded. Uniqueness and convergence of the limit solution are natural consequences. Besides, rate of convergence and exact solution in form of series expansion are obtained.

4.1 Algorithm 1

Both two interpretations of the probabilistic solution indicate stability. In the following analysis, only the iterative equation is used, while the probabilistic expression does not appear anymore.

On the image domain $\Omega=[0,M]\times[0, N]$, the equation for $u^{(n)}$ is (with boundary value zero)

(16) $$\Delta u^{(n)}-\lambda I_x^2u^{(n)}=\lambda(I_yv^{(n-1)}+I_t)I_x$$

As in Part Ⅰ [1], energy inequality of elliptic equation [1, 5] gives (where C arises from Friedrichs inequality [5])

(17) $$\iint_\Omega\left|u^{(n)}\right|^2dxdy\le C^2\lambda^2\iint_\Omega\left|(I_yv^{(n-1)}+I_t)I_x\right|^2dxdy,\ C=\min(M,N).$$

By linearity, $\Delta u(u^{(n)}-u^{(n-1)})-\lambda I^2_x(u^{(n)}-u^{(n-1)})=\lambda I_xI_y(v^{(n-1)}-v^{(n-2)})$, from this we have, by similar argument as above

(18) $$\begin{aligned}\iint_\Omega\left|u^{(n)}-u^{(n-1)}\right|^2dxdy&\le C^2\lambda^2\iint_\Omega\left|I_xI_y(v^{(n-1)}-v^{(n-2)})\right|^2dxdy\\&\le C^2C_{12}^2(I)\lambda^2\iint_\Omega\left|v^{(n-1)}-v^{(n-2)}\right|^2dxdy\end{aligned}$$

where Schwarz inequality is used in the second step, and

(19) $$C_{12}^2(I)=\iint_\Omega|I_xI_y|^2dxdy\ .$$

For digital images, the gray levels are finite, so is $C_{12}(I)$. A stronger argument comes from Theorem 3 of Part Ⅰ [1]; for generic images, $C_{12}(I)>0$, *i.e*. each image is not flat everywhere. Using L^2 norms, the above inequality can be rewritten as

(20) $$\left\|u^{(n)}-u^{(n-1)}\right\|_2\le CC_{12}(I)\lambda\left\|v^{(n-1)}-v^{(n-2)}\right\|_2.$$

Similarly,

(21) $$\left\|v^{(n)}-v^{(n-1)}\right\|_2\le CC_{12}(I)\lambda\left\|u^{(n-1)}-u^{(n-2)}\right\|_2.$$

So

(22) $$\left\|u^{(n)}-u^{(n-1)}\right\|_2\le(CC_{12}(I)\lambda)^2\left\|u^{(n-2)}-u^{(n-3)}\right\|_2.$$

Consequently,

(23) $$\sum_{n=1}^{\infty}\left\|u^{(n)}-u^{(n-1)}\right\|_2\le(CC_{12}(I)\lambda)^2\sum_{n=1}^{\infty}\left\|u^{(n)}-u^{(n-1)}\right\|_2+\left\|u^{(2)}-u^{(1)}\right\|_2+\left\|u^{(1)}-u^{(0)}\right\|_2.$$

Therefore, when $CC_{12}(I)\lambda < 1$, we have

$$(24)\quad \sum_{n=1}^{\infty}\left\|u^{(n)}-u^{(n-1)}\right\|_2 \le \frac{1}{1-(CC_{12}(I)\lambda)^2}\left[\left\|u^{(2)}-u^{(1)}\right\|_2+\left\|u^{(1)}-u^{(0)}\right\|_2\right]<\infty.$$

From (24) we know that the function sequence

$$(25)\quad u^{(n)} = u^{(0)} + \sum_{k=1}^{n}(u^{(n)} - u^{(n-1)}), n = 1,2,...$$

on Ω is convergent in L^2. Denote its limit function as u^*, also denote $\alpha=(CC_{12}(I)\lambda)^2$, we have the following error estimate

$$\left\|u^{(n)}-u^*\right\|_2 = \left\|\sum_{k=n+1}^{\infty}(u^{(k)}-u^{(k-1)})\right\|_2 \le \sum_{k=n+1}^{\infty}\left\|u^{(k)}-u^{(k-1)}\right\|_2$$

$$\le \sum_{m=\left[\frac{n}{2}\right]}^{\infty}\left\|u^{(2m)}-u^{(2m-1)}\right\|_2 + \sum_{m=\left[\frac{n}{2}\right]}^{\infty}\left\|u^{(2m+1)}-u^{(2m)}\right\|_2$$

$$(26)\quad \le \sum_{m=\left[\frac{n}{2}\right]}^{\infty}\alpha^{m-1}\left\|u^{(2)}-u^{(1)}\right\|_2 + \sum_{m=\left[\frac{n}{2}\right]}^{\infty}\alpha^{m}\left\|u^{(1)}-u^{(0)}\right\|_2$$

$$\le \sum_{m=\left[\frac{n}{2}\right]}^{\infty}\alpha^{m-1}\left\|u^{(2)}-u^{(1)}\right\|_2 + \sum_{m=\left[\frac{n}{2}\right]}^{\infty}\alpha^{m}\left\|u^{(1)}-u^{(0)}\right\|_2$$

$$\le \frac{1}{\alpha(1-\alpha)}\left[\left\|u^{(2)}-u^{(1)}\right\|_2+\alpha\left\|u^{(1)}-u^{(0)}\right\|_2\right]\alpha^{\left[\frac{n}{2}\right]},\ n=1,2,...$$

which implies that the algorithm is not only *convergent* in L^2 but *globally exponential stable* also, besides, its unique limit satisfies the Horn-Schunck system. From Theorem 3 in Part Ⅰ [1], the solution is infinitely smooth, so the unique L^2 limit equals to the exact solution everywhere.

Exponential convergence of the approximate solutions enables that we can view this alternating iteration as a *coarse-to-fine algorithm*.

Although the unique limit is independent of initial guess of optical flow, for purpose of faster convergence, there is a particular choice as follows, as a consequence of qualitative observation made in Part Ⅰ, *i.e.*

$$(27)\quad \begin{pmatrix} u^{(0)} \\ v^{(0)} \end{pmatrix} = -\frac{I_t}{I_x^2+I_y^2}\begin{pmatrix} I_x \\ I_y \end{pmatrix}$$

This is exactly the component paralleling to image gradient and given directly by considering only the fundamental constraint. To satisfy the standing assumption, the initial values on the boundary are forced to be zero.

4.2 Algorithm 2

Similarly, for system defined on image domain $\Omega=[0,M]\times[0,N]$ (with zero boundary values)

$$(28)\quad \begin{aligned} \Delta u^{(n)} &= \lambda(I_x u^{(n-1)} + I_y v^{(n-1)} + I_t)I_x \\ \Delta v^{(n)} &= \lambda(I_x u^{(n-1)} + I_y v^{(n-1)} + I_t)I_y \end{aligned}$$

two energy inequalities can be derived

$$(29)\quad \begin{aligned} \iint_\Omega\left|u^{(n)}-u^{(n-1)}\right|^2 dxdy &\le C^2\lambda^2\iint_\Omega\left|I_x^2(u^{(n-1)}-u^{(n-2)})+I_xI_y(v^{(n-1)}-v^{(n-2)})\right|^2 dxdy \\ \iint_\Omega\left|v^{(n)}-v^{(n-1)}\right|^2 dxdy &\le C^2\lambda^2\iint_\Omega\left|I_xI_y(u^{(n-1)}-u^{(n-2)})+I_y^2(v^{(n-1)}-v^{(n-2)})\right|^2 dxdy \end{aligned}$$

where the constant C is as in [17]. Denote C^2_{12}, C^2_{11}, C^2_{22} being integral of $|I_xI_y|^2$, $|I_x|^4$, $|I_y|^4$ on Ω, respectively (dependence on I is omitted), an application of Schwarz inequality yields

$$(30)\quad \begin{aligned} \left\|u^{(n)}-u^{(n-1)}\right\|_2 &\le C\lambda\left[C_{11}\left\|u^{(n-1)}-u^{(n-2)}\right\|_2+C_{12}\left\|v^{(n-1)}-v^{(n-2)}\right\|_2\right] \\ \left\|v^{(n)}-v^{(n-1)}\right\|_2 &\le C\lambda\left[C_{12}\left\|u^{(n-1)}-u^{(n-2)}\right\|_2+C_{22}\left\|v^{(n-1)}-v^{(n-2)}\right\|_2\right] \end{aligned}$$

In matrix notation, this means

$$(31)\quad \begin{pmatrix}\left\|u^{(n)}-u^{(n-1)}\right\|_2 \\ \left\|v^{(n)}-v^{(n-1)}\right\|_2\end{pmatrix} \le C\lambda\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22}\end{pmatrix}\begin{pmatrix}\left\|u^{(n-1)}-u^{(n-2)}\right\|_2 \\ \left\|v^{(n-1)}-v^{(n-2)}\right\|_2\end{pmatrix} \triangleq A\begin{pmatrix}\left\|u^{(n-1)}-u^{(n-2)}\right\|_2 \\ \left\|v^{(n-1)}-v^{(n-2)}\right\|_2\end{pmatrix}.$$

The eigenvalues of A are

$$(32)\quad s_1 = C\lambda\left[C_{11}+C_{22}+\sqrt{(C_{11}-C_{22})^2+4C_{12}^2}\right],$$

and

$$(33)\quad s_2 = C\lambda\left[C_{11}+C_{22}-\sqrt{(C_{11}-C_{22})^2+4C_{12}^2}\right].$$

Both eigenvalues are nonnegative since $C^2_{12}\leqslant C_{11}C_{22}$ by Schwarz inequality; furthermore, $s_1=0$ means that I is flat everywhere, while $s_2=0$ amounts to that I is flat along x axis, or I is flat along y axis, or $|I_x|$ and $|I_y|$ linearly correlate. Obviously, all these possibilities are not generic. Therefore, for generic images, both eigenvalues are positive, so when λ is small enough, both eigenvalues are less than 1 and the algorithm globally exponential converges in L^2. In the non-generic cases, the eigenvalues will be smaller and the conclusion still holds.

Similar comments apply as in the previous subsection.

5. Conclusions and Discussions

Two decoupled form of alternating iteration algorithms are proposed to solve the Horn-Schunck system though the later is coupled *per se*. Analytical solutions for each single iteration are explicitly given via several standard techniques from theory of scalar elliptic equations and complex analysis. These algorithms are obviously consistent. Exponential stability is established using Banach fixed point principle for contraction mappings.

One may criticize that in both algorithms PDEs need to be solved one after another thus they do not have practical merits. Things are not that bad, since by exponential convergence, only a small number of PDEs are needed to be solved in practice, moreover, if we code these analytical formulae, they can become true algorithms. This has been completed in a companioned paper. Another flaw of these algorithms comes from the fact that these algorithms converge only for small relevant parameters, especially for small λ. The estimates obtained are surely conservative.

After this paper had been completed, the contributions [9] and [10] suddenly came into our view and has since greatly encouraged us. We would like to make some comments. The following comments will involve all our three papers of this series.

Both [9] and [10] concerned the convergence issue of the Horn-Schunck algorithm and obtained similar positive results as ours, i.e. the classical algorithm is exponentially convergent, however, several differences exist between theirs and ours:

1) They concerned only the performance of the original classical algorithm while we studied both that algorithm and the Horn-Schunck system itself (including its well-posedness, its decouplability, and the relationship among the system, the original variational formulation, and the discrete algorithm);

2) Their methods are both purely algebraic, while we used both algebraic and analytic methods;

3) We have carried out a lot of experiments which validated our assertions while they did not;

4) We proposed two alternating iteration algorithms and studied their performances while they did not;

5) We discussed the regularity issue both about the solution of the Horn-Schunck system and about natural images while they did not care.

Acknowledgements

We sincerely thank the editors and all anonymous reviewers for kind suggestions on both the structure and presentation which greatly improved this paper.

This research is supported by National Science Fund under Grant No. NSF60835005 and NSF90820302, and supported by 973 Plan Fund under Grant No. 2007CB311001.

REFERENCES

[1] Dong G. H., An X. J., Fang Y. Q., Hu D. W., Horn-Schunck optical flow equations.Part Ⅰ: Stability and rate of convergence of the classical algorithm (accepted by Journal of Central South University, under minor modification)
[2] Dong G. H., An X. J., Hu D. W., Horn-Schunck optical flow equations. Part Ⅱ: Decoupling via linear transformation (accepted by IScIDE 2012, Nanjing, to appear in LNCS)
[3] Horn B. K. P., Schunck B. G., Determining optical flow, AI 17 (1981), 185-203
[4] Wu L. D., Computer Vision, Fudan University Press, Shanghai (1993, in Chinese)
[5] Gu C. H., Li T. T., Chen X. X., Shen W. X., Qin T. H., Shi J. H., Methods of Mathematical Physics, Fudan University Press, Shanghai (1985, in Chinese)
[6] Courant R., Hilbert D., Methods of Mathematical Physics, Vol. Ⅱ, Interscience, New York (1962)
[7] Gong G. L., Stochastic Differential Equations and Applications, Tsinghua University Press, Beijing. (2008, in Chinese)
[8] Stein E. M., Shakarchi R., Complex Analysis, Princeton Lectures in Analysis Ⅱ, Princeton University Press (2003)
[9] Mitiche A., Mansouri A.-R., On convergence of the Horn and Schunck optical-flow estimation method, IEEE Transactions on Image Processing, 13 (2004), No. 6, 848-852
[10] Kamedal Y., Imiya A., Ohnishi N., A convergence proof for the Horn-Schunck optical-flow computation scheme using neighborhood decomposition, In Brimkov V. E., Barneva R.P., Hauptman H. A. (Eds.): IWCIA 2008, LNCS 4958, 262–273 (2008)
[11]Jianxue Chen, Yan Luo, The Comparison of Entry Mode on Transnational Corporation in NIE, *Advanced Management Science*, 1 (2011), No.1, 22-27

***Authors**: Guohua Dong, Xiangjing An, Dewen Hu are with the College of Mechatronics and Automation, National University of Defense Technology, Changsha, Hunan, China.*
E-mail: ghdong@nudt.edu.cn.