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Impedance of an isolated rectangular conductor

Abstract. *In this paper new analytical numerical method for calculating of impedance of any long straight isolated conductor of rectangular cross section is presented. The conductor is divided into a great number of rectangular subconductors of very small size. Self inductance of each subconductor and mutual inductance between them are calculated using the special function obtained from an integral equation. All resistances and inductances of subconductors compose the impedance matrix hence the admittance matrix is determined. Consequently, the impedance of whole rectangular conductor is calculated. This impedance takes into account skin effect.*

Streszczenie. *W pracy przedstawia się nową analityczno-numeryczna metodę wyznaczania impedancji pojedynczego przewodu prostokątnego o dowolnej długości. Przewód dzieli się na wiele przewodów elementarnych o bardzo małych wymiarach. Indukcyjności własne każdego z nich jak również indukcyjności wzajemne między nimi oblicza się za pomocą specjalnej funkcji otrzymane z równania całkowego. Wszystkie rezystancje i indukcyjności przewodów elementarnych tworzą macierz impedancji skąd wyznacza się macierz admitancji. W konsekwencji oblicza się impedancję całego przewodu prostokątnego. Impedancja ta uwzględnia zjawisko naskórkowości.(**Impedancja pojedynczego przewodu o przekroju prostokątnym**)*

Key words: rectangular busbar, self inductance, mutual inductance, impedance
Słowa kluczowe: prostokątny przewód szynowy, indukcyjność własna, indukcyjność wzajemna, impedancja

Introduction

Real lumped isolated conductor can be modeled as an impedance which plays an important role not only in power circuits [1], but also in printed circuit board (PCB) lands [2, 3]. For large dimensions of conductor or/and high frequency skin effect becomes significant. The impedance of conductor

(1) $$Z = R + \mathrm{j}\,\omega\, L$$

depends on conductivity, dimensions and frequency. ($\omega = 2\,\pi\, f$). Many connectors contain pins with rectangular cross section. Such geometries cannot be easily handled through analytical techniques and as a consequence of that various numerical calculation methods for computing the high frequency and low frequency impedance of isolated conductor are applied. In general these methods consist in dividing the conductor into parallel long segments having small, rectangular cross sections assumed to have their length much greater (at least five times) than their thickness and width [4-11]. That is why in this paper an analytical numerical method for calculating of impedance of any length rectangular conductor is proposed.

In order to determine the self impedance of the rectangular busbar we should calculate its resistance and self inductance which depends on dimensions of busbar and on frequency *f*.

DC resistance and self inductance

If a conductor has a constant cross-sectional area *S* along its length and in case of DC, low frequency (for busbars of dimensions used in electrical power distribution system [1]) or for a very thin strip conductor (in printed circuit board [2, 3]) we can assume that the current density is constant and given as $\underline{J}(X) = \underline{I}\,/\,S$ and then the self resistance of a rectangular busbar (Fig. 1) is

(2) $$R_0 = \frac{l}{\sigma\, a\, b}$$

and its self inductance [12, 13]

(3) $$L_0 = \frac{\mu_0}{4\pi\ a^2\, b^2}\int_v\int_v \frac{1}{\rho_{XY}}\mathrm{d}\,v\,\mathrm{d}\,v = \frac{\mu_0}{4\,\pi}\frac{1}{a^2\, b^2}F$$

where

(4) $$F = \int_0^l\int_0^l\int_0^b\int_0^b\int_0^a\int_0^a \frac{1}{\rho_{XY}}\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z_1\mathrm{d}z_2$$

is a sixtuple definite integral of six variables $(x_1, x_2, y_1, y_2, z_1, z_2)$. In (2) and (3) σ is conductivity, ω is angular velocity, $\underline{J}(Y)$ is the complex current density at source point $Y = Y(x_2, y_2, z_2)$, $\underline{J}(X)$ is the complex current density at point of observation $X = X(x_1, y_1, z_1)$, v is conductor's volume and $\rho_{XY} = \sqrt{r_{XY}^2 + (z_2 - z_1)^2}$ is distance between the point of observation X and the source point Y (Fig.1), where $r_{XY} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

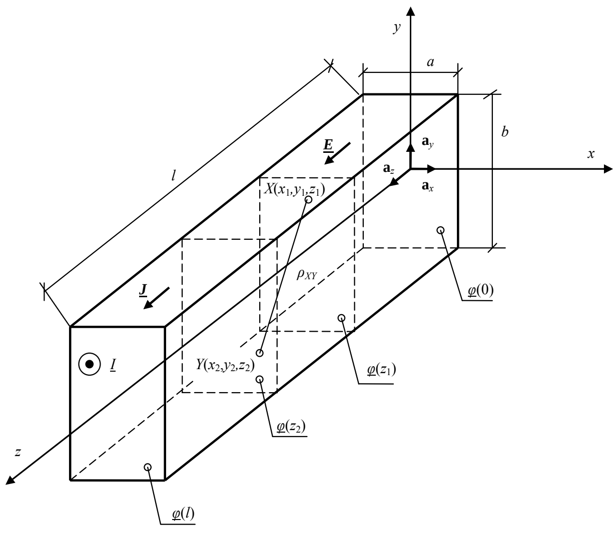

Fig. 1. Isolated conductor of rectangular cross section with thickness *a*, width *b*, length *l* and current $\underline{I}$

In general case this integral (3) is very difficult to calculate. But in our case we can put $x = x_2 - x_1$, $y = y_2 - y_1$, $z = z_2 - z_1$ and first calculate a sixtuple indefinite integral

(5)
$$F(x,y,z)=\iiiiiint \frac{\mathrm{d}x\mathrm{d}x\mathrm{d}y\mathrm{d}y\mathrm{d}z\mathrm{d}z}{\rho_{XY}(x,y,z)}=\frac{1}{72}\left\{\begin{array}{l}\frac{6}{5}\left(x^4+y^4+z^4-3x^2y^2-3x^2z^2-3y^2z^2\right)\sqrt{x^2+y^2+z^2}-\\ -12xyz\left(z^2\tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}}+y^2\tan^{-1}\frac{xz}{y\sqrt{x^2+y^2+z^2}}+x^2\tan^{-1}\frac{yz}{x\sqrt{x^2+y^2+z^2}}\right)-\\ -3x(y^4-6y^2z^2+z^4)\ln\left(x+\sqrt{x^2+y^2+z^2}\right)-3y(x^4-6x^2z^2+z^4)\ln\left(y+\sqrt{x^2+y^2+z^2}\right)-\\ -3z(x^4-6x^2y^2+y^4)\ln\left(z+\sqrt{x^2+y^2+z^2}\right)\end{array}\right\}$$

Hence the DC self inductance of the conductor of rectangular cross section is given by following formula

(6)
$$L_0=\frac{\mu_0}{4\pi}\frac{1}{a^2\,b^2}\left[\left[\left[F(x,y,z)\right]_{0,0}^{a,-a}(x)\right]_{0,0}^{b,-b}(y)\right]_{0,0}^{l,-l}(z)=\frac{\mu_0}{4\pi}\frac{1}{a^2\,b^2}\sum_{i=1}^{i=4}\sum_{j=1}^{j=4}\sum_{k=1}^{k=4}\left(-1\right)^{i+j+k+1}F(p_i,q_j,r_k)$$

DC mutual inductance

We consider a general case of two parallel conductors of rectangular cross section shown in Fig. 2. The positions of conductors are determined by coordinates of diagonal corner points: (s_1,s_5,s_9), (s_2,s_6,s_{10}) of the first wire of dimensions $a_1\times b_1\times l_1$ and (s_3,s_7,s_{11}), (s_4,s_8,s_{12}) of the second wire of dimensions $a_2\times b_2\times l_2$. We assume the conductors to be parallel.

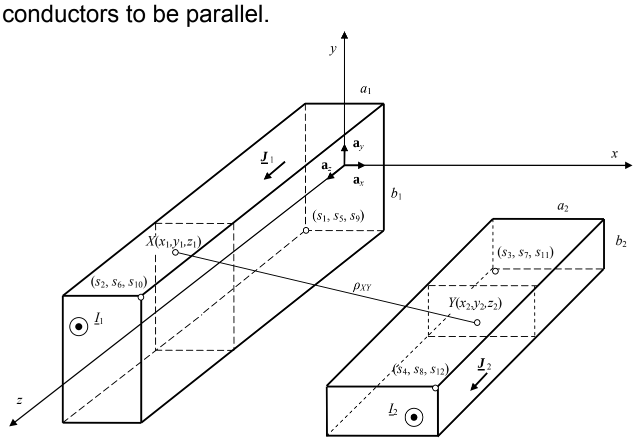

Fig. 2. Two parallel conductors of rectangular cross section with currents $\underline{I}_1$ and $\underline{I}_2$

If conductors have a constant cross-sectional area S_1 and S_2 along theirs lengths, in case of DC, low frequency or for a thin strip conductors we can assume that the current density is constant and given as $\underline{J}_{11}(X)=\underline{I}_1/S_1$ and $\underline{J}_{22}(X)=\underline{I}_2/S_2$ then the DC mutual inductance between two straight parallel conductors [13,14]

(7)
$$\begin{aligned}M_0&=\frac{\mu_0}{4\pi}\frac{1}{a_1\,a_2\,b_1\,b_2}\times\\ &\left[\left[\left[F(x,y,z)\right]_{s_1-s_3,\,s_2-s_4}^{s_1-s_4,\,s_2-s_3}(x)\right]_{s_5-s_7,\,s_6-s_8}^{s_5-s_8,\,s_6-s_7}(y)\right]_{s_9-s_{11},\,s_{10}-s_{12}}^{s_9-s_{12},\,s_{10}-s_{11}}(y)=\\ &=\frac{\mu_0}{4\pi}\frac{1}{a_1\,a_2\,b_1\,b_2}\left[\left[\left[F(x,y,z)\right]_{p_2,p_4}^{p_1,p_3}(x)\right]_{q_2,q_4}^{q_1,q_3}(y)\right]_{r_2,r_4}^{r_1,r_3}(z)=\\ &=\frac{\mu_0}{4\pi}\frac{1}{a_1\,a_2\,b_1\,b_2}\sum_{i=1}^{i=4}\sum_{j=1}^{j=4}\sum_{k=1}^{k=4}\left(-1\right)^{i+j+k+1}F(p_i,q_j,r_k)\end{aligned}$$

where

(8)
$$F=\int_{s_{11}}^{s_{12}}\int_{s_9}^{s_{10}}\int_{s_7}^{s_8}\int_{s_5}^{s_6}\int_{s_3}^{s_4}\int_{s_1}^{s_2}\frac{1}{\rho_{XY}}\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2\mathrm{d}z_1\mathrm{d}z_2$$

$a_1=s_2-s_1$, $b_1=s_6-s_5$, $a_2=s_4-s_3$ and $b_2=s_8-s_7$. Also $l_1=s_{10}-s_9$ and $l_2=s_{12}-s_{11}$.

Case of skin effect

The rectangular conductor is divided into the same subconductors of length *l* and much smaller sections with thickness $\Delta a={}^{a}\!/_{N_a}$ and width $\Delta b={}^{b}\!/_{N_b}$ (Fig. 3) where N_a is the number of divisions along the conductor thickness and N_b is the number of divisions along conductor width. The total number of subconductors equals $N_c=N_aN_b$. The numbers of subconductors are: $m=(i-1)N_a+k$ and $n=(j-1)N_a+l$ for $i,j=1,2,...,N_b$ and $k,l=1,2,...,N_a$.

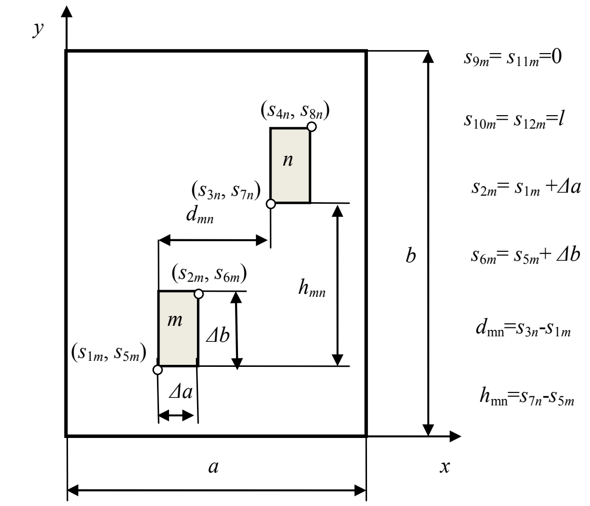

Fig. 3. The rectangular conductor divided into subconductors

By assuming that the subconductor current densities are uniform, as in case of DC current, its self impedance (for $m=n$) is

(9)
$$\underline{Z}_{mm}=\underline{Z}_{nn}=R_0+\mathrm{j}\,\omega\,L_0$$

for $a=\Delta a$ and $b=\Delta b$ into (6) and (7). The mutual impedance (for $m\neq n$) between the m^{th} and n^{th} subconductor is given by following formula

(10)
$$\underline{Z}_{mn}=\underline{Z}_{nm}=\mathrm{j}\,\omega\,M_0$$

for $a_1 = a_2 = a = \Delta a$ and $b_1 = b_2 = b = \Delta b$ into (5) and (6).

The voltage drop across the rectangular conductor is $\underline{U}$, the drop across each of the subconductors is also $\underline{U}$. So for the subconductor in the i^{th} row and j^{th} column (or across of n^{th} subconductor), this voltage is

(11) $$\underline{U}_n = \underline{U} = \sum_{m=1}^{N_c} \underline{Z}_{nm} \underline{I}_m$$

Hence the currents in each of the subconductors are

(12) $$\underline{I}_n = \sum_{m=1}^{N_c} \underline{Y}_{nm} \underline{U}_n$$

where $\underline{Y}_{ij}$ is the element of the admittance matrix.

All the subconductors are in parallel and once the current vector is known, the total current through the rectangular conductor is

(13) $$\underline{I} = \sum_{m=1}^{N_c} \underline{I}_m = \underline{Y}\underline{U}$$

where the admittance

(14) $$\underline{Y} = \sum_{m=1}^{N_a} \sum_{n=1}^{N_b} \underline{Y}_{mn}$$

And finally the impedance of the rectangular conductor is

(15) $$\underline{Z} = \underline{Y}^{-1} = R + \mathrm{j}\,\omega L$$

where R and L are its resistance and its self inductance at the frequency f.

Computational results

To compute the impedance of the rectangular conductor a numerical program was written. Given a, b, l, N_a, N_b and f computes the frequency dependent resistance and self inductance of this conductor (Table 1 and Table 2) using the previously method.

Table 1. The frequency dependent resistance of rectangular conductor

f	R/R_0			
kHz	$b:a=1$	$b:a=5$	$b:a=10$	$b:a=20$
0.01	1.000007	1.000232	1.000982	1.003861
0.1	1.000735	1.021946	1.074437	1.167352
1.0	1.069717	1.525194	1.726961	1.903913
10	2.471251	4.213178	4.930567	5.548998
100	6.724710	9.372228	11.040150	12.454146
1000	13.25425	10.768692	12.735175	14.587965

Table 2. The frequency dependent inductance of rectangular conductor

f	L/L_0			
kHz	$b:a=1$	$b:a=5$	$b:a=10$	$b:a=20$
0.01	1.000013	0.990512	0.991035	0.992034
0.1	0.999997	0.989318	0.989318	0.978025
1.0	0.998455	0.971508	0.971508	0.960799
10	0.974501	0.954460	0.953466	0.953033
100	0.960187	0.949479	0.949479	0.950358
1000	0.956946	0.949096	0.949096	0.950101

Conclusions

The impedance calculation in high frequencies is very important. In this paper we have presented a new method for calculating of impedance of an isolated conductor of rectangular cross section of any dimensions and length. Our method is very fast and simple with accurate results for straight rectangular conductors.

The resistance and self inductance of rectangular conductor depend on its conductivity, dimensions and frequency. At relatively low frequency the resistance and self inductance values are nearly identical. At high frequency the skin effect is significant and resistance rises fast above DC resistance. The self inductance decreases slightly faster than resistance.

Our method is analytically simple and can also replace the traditional Dwight's curves.

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