Janusz WALCZAK[1], Agnieszka JAKUBOWSKA[1]

Silesian University of Technology (1)

Resonance in series fractional order *RL$_\beta$C$_\alpha$* circuit

***Abstract**. The paper describes the results of studies on the phase and magnitude resonance phenomenon in a series RLC circuit with fractional order reactive elements. Formulas for frequency characteristics and resonance conditions have been derived. Simulations of concerned fractional order system have been conducted too.*

***Streszczenie.** W artykule opisano wyniki badań zjawiska rezonansu w szeregowym obwodzie RLC z elementami reaktancyjnymi L_β, C_α ułamkowego rzędu. Wyprowadzono zależności określające charakterystyki częstotliwościowe układu oraz warunki rezonansu fazy i amplitudy. Podano także wyniki badań symulacyjnych rozpatrywanego układu. (**Rezonans fazy i amplitudy w szeregowym obwodzie RL$_\beta$C$_\alpha$ ułamkowego rzędu**).*

Keywords: phase and magnitude resonance, fractional order inductance and capacitance.
Słowa kluczowe: rezonans fazowy i amplitudowy, indukcyjność i pojemność ułamkowego rzędu.

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Introduction

Fractional order elements L_β, C_α represent a generalization of classic reactive elements LC [2]. Their mathematical models in frequency domain are frequently described by relations [1], [2]:

(1) $$Z_L(j\omega) = R_L + (j\omega)^\beta L, \qquad \beta \in R^+,$$

(2) $$Z_C(j\omega) = R_C + (j\omega)^{-\alpha} C^{-1}, \qquad \alpha \in R^+.$$

where: R_L, R_C– internal series resistances, L, C – nominal inductance and capacitance, α, β – fractional order coefficients.

Many practical realizations of these elements are known [2], and supercapacitors are one of the best known implementation of the fractional order elements. Fractional order elements find various applications in electrical engineering, electronics and control theory [2], [3]. Properties of systems containing fractional order elements differ from those of systems with classic RLC elements. For instance, basic elements L_β, C_α have identical properties as LTI two - terminal circuits class ± RLC, dependent on α and β coefficient values. Relations (1) and (2) can be written as:

(3) $$Z_L(j\omega) = R_L + \omega^\beta L \cos\left(\frac{\pi}{2}\beta\right) + j\omega^\beta L \sin\left(\frac{\pi}{2}\beta\right),$$

(4) $$Z_C(j\omega) = R_C + \omega^{-\alpha} C^{-1} \cos\left(\frac{\pi}{2}\alpha\right) - j\omega^{-\alpha} C^{-1} \sin\left(\frac{\pi}{2}\alpha\right).$$

Mentioned features of their impedance arise from the analysis of relations (3) and (4). It is illustrated in Fig. 1.

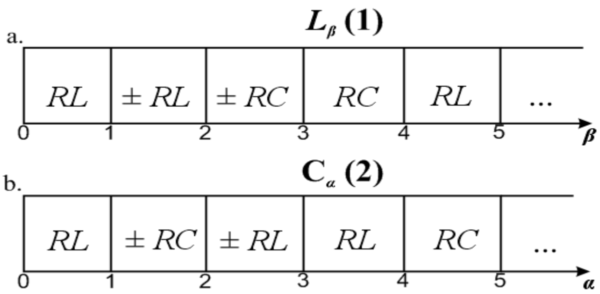

Fig. 1. The evolution of elements $L_\beta C_\alpha$ features in dependence of parameters β, α values:
a. Element L_β (formula (1)),
b. Element C_α (formula (2)).

Research on fractional order systems is conducted in various directions [2], [4], [5]. One of them concerns the analysis of fractional order system features in frequency domain [2], [6]. Studies of the resonance phenomena in a series RLC_α circuit were presented in articles [6], [7]. This article is its continuation and concerns the analysis of phase and magnitude resonance in a series $RL_\beta C_\alpha$ circuit.

Frequency model of the system

The considered $RL_\beta C_\alpha$ model is shown in Fig. 2. It consists of a fractional coil (inductor) L_β and a fractional capacitor C_α (e.g. supercapacitor), which impedances are described by relations (1) and (2) respectively.

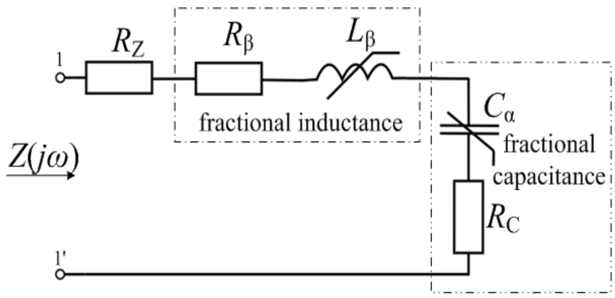

Fig.2. Series $RL_\beta C_\alpha$ circuit

The impedance of the circuit (Fig. 2) seen from the terminals 1 -1' is represented by:

(5) $$\begin{aligned} Z(j\omega) &= \left(R + (j\omega)^\beta L + (j\omega)^{-\alpha} C^{-1}\right) = \\ &= \left(R + \omega^\beta L \cos\left(\frac{\pi}{2}\beta\right) + \omega^{-\alpha} C^{-1} \cos\left(\frac{\pi}{2}\alpha\right)\right) + \\ &+ j\left(\omega^\beta L \sin\left(\frac{\pi}{2}\beta\right) - \omega^{-\alpha} C^{-1} \sin\left(\frac{\pi}{2}\alpha\right)\right) = \\ &= \operatorname{Re}\{Z(j\omega)\} + j \operatorname{Im}\{Z(j\omega)\} = \\ &= |Z(j\omega)| \cdot \exp(j\varphi(\omega)) \end{aligned}$$

where:

(6) $$R = R_Z + R_L + R_C.$$

(7) $$|Z(j\omega)| = \left[\left(R + \omega^\beta L \cos\left(\frac{\pi}{2}\beta\right) + \omega^{-\alpha} C^{-1} \cos\left(\frac{\pi}{2}\alpha\right)\right)^2 + \left(\omega^\beta L \sin\left(\frac{\pi}{2}\beta\right) - \omega^{-\alpha} C^{-1} \sin\left(\frac{\pi}{2}\alpha\right)\right)^2\right],$$

(8) $$\varphi(\omega) = arctg \frac{\omega^\beta L \sin\left(\frac{\pi}{2}\beta\right) - \omega^{-\alpha} C^{-1} \sin\left(\frac{\pi}{2}\alpha\right)}{R + \omega^\beta L \cos\left(\frac{\pi}{2}\beta\right) + \omega^{-\alpha} C^{-1} \cos\left(\frac{\pi}{2}\alpha\right)}$$

.

Exemplary graphs of the functions $|Z(j\omega)|$, $\varphi(\omega)$, $\mathrm{Re}\{Z(j\omega)\}$, $\mathrm{Im}\{Z(j\omega)\}$ defined by formulas (5), (7), (8) are shown in Fig. 3.

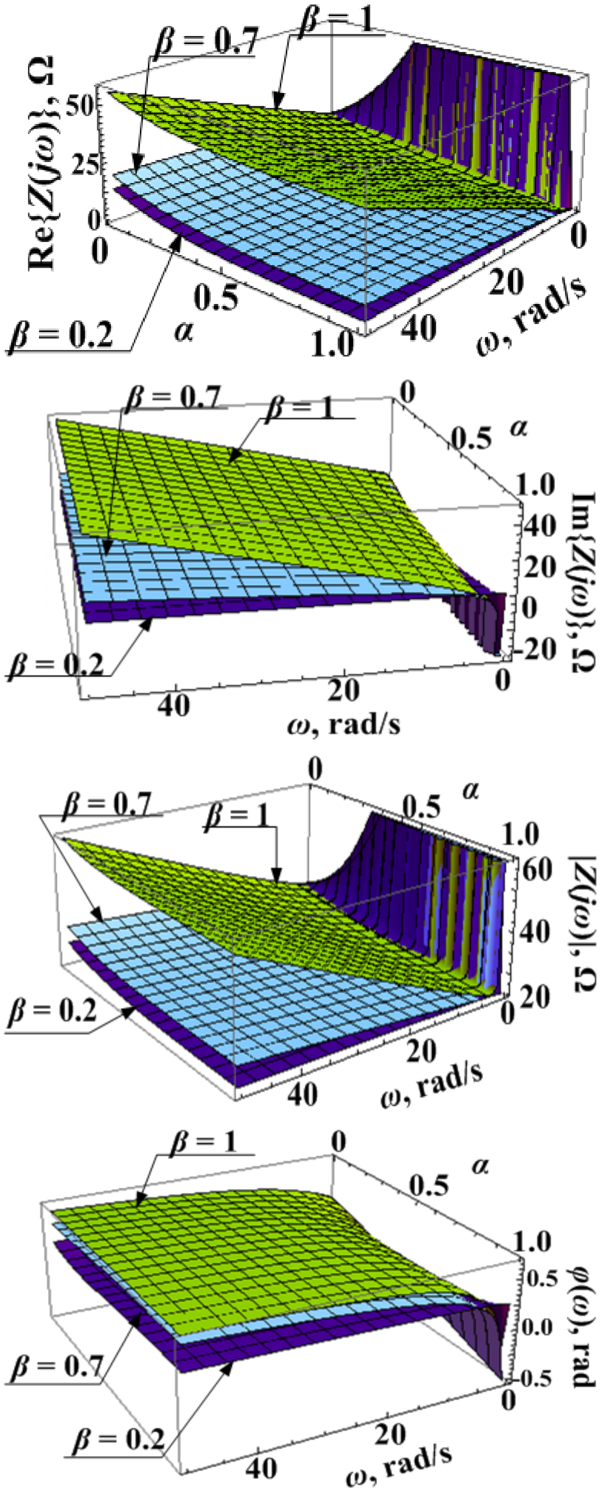

Fig. 3. Graphs of the functions for $RL_\beta C_\alpha$ system in dependence of parameters β, α values:

a. $\mathrm{Re}\{Z(j\omega)\}$,
b. $\mathrm{Im}\{Z(j\omega)\}$,
c. Impedance module $|Z(j\omega)|$
d. Impedance phase $\varphi(\omega)$.

Simulations have been performed for elements values: $R = 10\ \Omega$, $L = 1$ H and $C = 0.1$ F.

Phase resonance conditions

The formula (5) suggests, that the phase resonance conditions:

$$\mathrm{Im}\{Z(j\omega)\} = 0, \quad \mathrm{Im}\{Y(j\omega)\} = 0, \tag{9}$$

are the same. Hence, based on the formulas (5) and (9) there can be derived a relationship for the phase resonance angular frequency ω_{rp} in the system from Fig.2:

$$\omega_{rp} = \sqrt[\alpha+\beta]{\frac{1}{LC}\,\frac{\sin\left(\frac{\pi}{2}\alpha\right)}{\sin\left(\frac{\pi}{2}\beta\right)}}\,. \tag{10}$$

Formula (10) analysis shows that the resonance state does not exist for all α, β values of elements from the system from Fig. 2, comparing with Fig. 4.

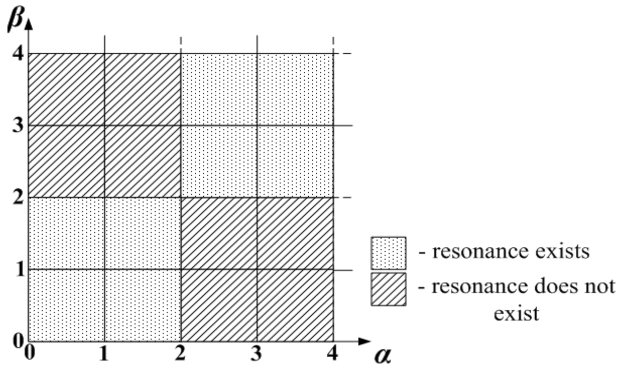

Fig.4. Conditions of phase resonance existence.

It can be notices, that in specific cases:

1. $\alpha = \beta$:

$$\omega_{rp} = \sqrt[2\alpha]{\frac{1}{LC}}\,, \tag{11}$$

2. $\alpha = \beta = 1$:

$$\omega_{rp} = \frac{1}{\sqrt{LC}}\,. \tag{12}$$

The case defined by formula (12) describes the classic resonance in a series RLC circuit. An exemplary resonance curve of voltages in the system from Fig. 2 with voltage source defined as:

$$u(t) = \sqrt{2}|u_0|\sin(\omega t), \tag{13}$$

is shown in Fig. 5:

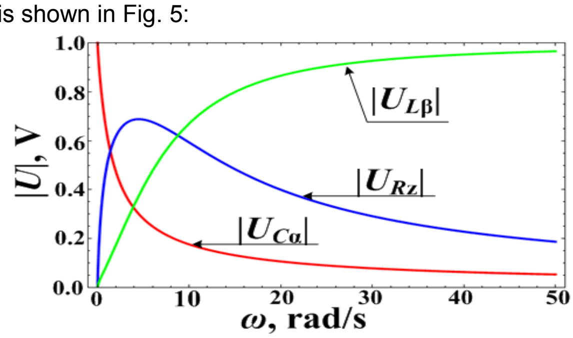

Fig.5. The resonance curve for the circuit from Fig. 2 for parameters: $\alpha = 0.75$, $\beta = 0.95$, $R = 10\ \Omega$.

In resonance state the circuit equivalent impedance $Z_r(j\omega)$ is defined by a formula:

(14) $$Z_r(j\omega) = R + 2\left(\frac{1}{LC}\frac{\sin\left(\frac{\pi}{2}\alpha\right)}{\sin\left(\frac{\pi}{2}\beta\right)}\right)^{\frac{\beta}{\alpha+\beta}} L\cos\left(\frac{\pi}{2}\beta\right),$$

which implies, that the total impedance can take either positive or negative values.

It is also easy to demonstrate that for a parallel $L_\beta C_\alpha$ circuit (for $R = 0$, compare with formula (5)) the resonance angular frequency ω'_{rp} is defined by relation (compare with (10)):

(15) $$\omega'_{rp} = \sqrt[\alpha+\beta]{\frac{1}{LC}\cdot\frac{\sin\left(\frac{\pi}{2}\beta\right)}{\sin\left(\frac{\pi}{2}\alpha\right)}}.$$

The resonance frequencies of series and parallel $L_\beta C_\alpha$ circuits differ from each other, in contrast to classic integer order LC circuits.

Magnitude resonance conditions

The RMS current value flowing in series $RL_\beta C_\alpha$ circuit, supplied by sinusoidal voltage source described by the formula (13) can be written as:

(16) $$|I(j\omega)| = \frac{|U_0|}{|Z(j\omega)|}.$$

For a circuit supplied by current source of RMS value $|I_0|$, the voltage across the system terminals 1 - 1' from Fig. 2 is described by relation:

(17) $$|U(j\omega)| = |I_0|\cdot|Z(j\omega)| = \frac{|I_0|}{|Y(j\omega)|},$$

where: $|Y(j\omega)|$ - admittance of a series $RL_\beta C_\alpha$ circuit.

Exemplary graphs of impedance and admittance module of the circuit from Fig. 2 are shown below. Simulations were performed for parameters values: $R = 10\ \Omega$, $L = 1$ H, $C = 0.1$ F, $\alpha = 0.75$ and $\beta = 0.95$.

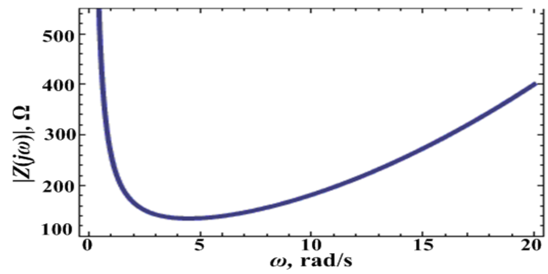

Fig.6. Graph of the function $|Z(j\omega)|$ (based on the formula (7)), for $|U_0| = 1$.

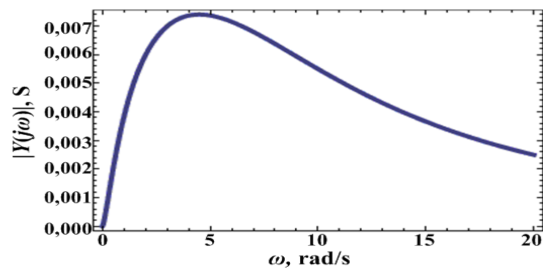

Fig.7. Graph of the function $|Y(j\omega)| = 1/\ |Z(j\omega)|$, for $|I_0| = 1$.

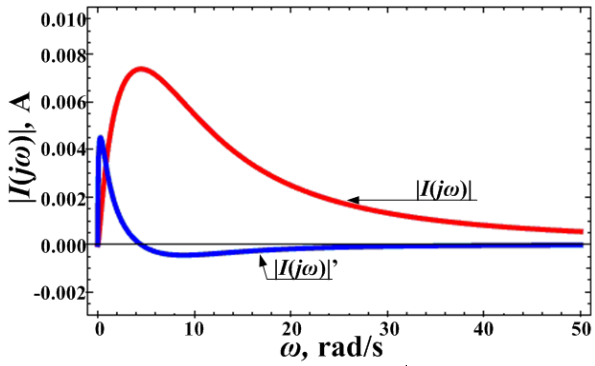

Fig.8. Graph of the function $|I(j\omega)|$ and its 1st derivative for $|U_0| = 1$.

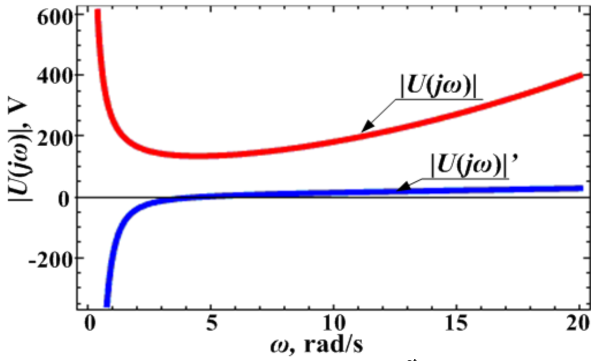

Fig.9. Graph of the function $|U(j\omega)|$ and its 1st derivative for $|I_0| = 1$.

The magnitude resonance condition for the circuit supplied by a voltage source $|U_0|$:

(18) $$\frac{\partial|I(j\omega)|}{\partial\omega} = 0,$$

as well as for the circuit supplied by a current source $|I_0|$:

(19) $$\frac{\partial|U(j\omega)|}{\partial\omega} = 0,$$

implies the following equation:

(20) $$\left[R + \omega^\beta L\cos\left(\frac{\pi}{2}\beta\right) + \frac{1}{\omega^\alpha C}\cos\left(\frac{\pi}{2}\alpha\right)\right]\cdot$$
$$\cdot\left[\beta\omega^{\beta-1}L\cos\left(\frac{\pi}{2}\beta\right) - \frac{\alpha}{\omega^{\alpha+1}C}\cos\left(\frac{\pi}{2}\alpha\right)\right] +$$
$$+\left[\omega^\beta L\sin\left(\frac{\pi}{2}\beta\right) - \frac{1}{\omega^\alpha C}\sin\left(\frac{\pi}{2}\alpha\right)\right]\cdot$$
$$\cdot\left[\beta\omega^{\beta-1}L\sin\left(\frac{\pi}{2}\beta\right) + \frac{\alpha}{\omega^{\alpha+1}C}\sin\left(\frac{\pi}{2}\alpha\right)\right] = 0.$$

It can be proved that the magnitude resonance conditions for current and voltage are the same and they occur at the same frequency for a given series $RL_\beta C_\alpha$ circuit.

In a particular case, for $R = 0$, the magnitude resonance angular frequency ω_{rm} of the $RL_\beta C_\alpha$ circuit with voltage and current supplying is given by the equation (compare with formulas (10) and (15)):

(21) $$\omega_{rm} = \sqrt[\alpha+\beta]{\frac{1}{2\beta LC}\left((\alpha-\beta)\cos(\alpha+\beta) + \sqrt{(\beta-\alpha)^2\cdot\cos(\alpha+\beta) + 4\alpha\cdot\beta}\right)}.$$

In specific cases, the magnitude resonance angular frequency is (see also formulas (11) and (12)):

3. $\alpha = \beta$:

(22) $$\omega_{rm} = \sqrt[2\alpha]{\frac{1}{LC}},$$

4. $\alpha = \beta = 1$:

(23) $$\omega_{rm} = \frac{1}{\sqrt{LC}}.$$

In cases described by formulas (22) and (23) the phase and magnitude resonance occur simultaneously at the same frequency.

Determination of the parameters α and β values for which the magnitude resonance exists is not simple, as in the case of phase resonance (see Fig. 4). Fig. 11 shows the magnitude resonance existence conditions for parameters α and β evolution.

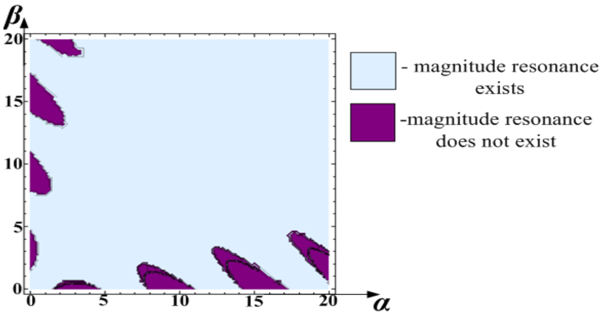

Fig.11. Conditions of magnitude resonance existence.

An exemplary dependence of the magnitude resonance angular frequency as a function of coefficients α and β evolution is shown in Fig. 10.

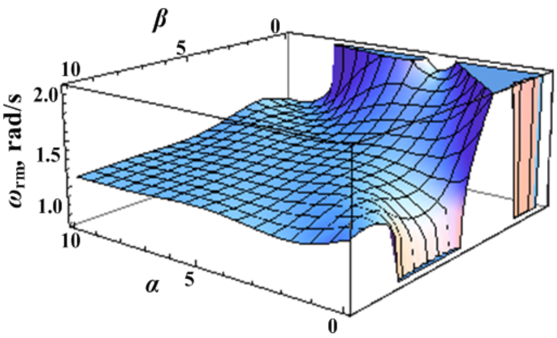

Fig.10. The magnitude resonance frequency for fractional order coefficients α and β evolution for parameters: L = 1H, C = 0.1 F.

Generally, for $R \neq 0$, equation (20) can be determined only numerically. Exemplary characteristics of the magnitude resonance angular frequency ω_{rm} as a function of coefficient α, determined by simulations, are presented in Fig. 12.

Fig. 12 shows, that the coefficient β has an impact on the shape of the resonance angular frequency ω_{rm} curve. For small values of β there is no explicit maximum of magnitude resonance angular frequency ω_{rm}. As β grows up to 0.5, a clear maximum of ω_{rm} appears and ω_{rm} takes higher values. For increasing parameter β maximum of ω_{rm} shifts toward lower values of α and its value decreases significantly. The value of β for which ω_{rm} reaches the local maximum (in range (0,1)) should be calculated by adopting the equation (20) as a function of single variable β and counting its first and second derivatives with respect to β.

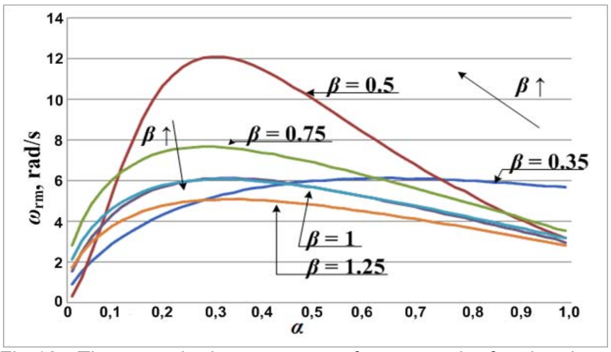

Fig.12. The magnitude resonance frequency in fractional order coefficient α evolution for parameters: R = 10 Ω, L = 1H, C = 0.1 F and chosen values of coefficient β

Conclusions

The paper presents an analysis of the phase and magnitude resonance effect in a series $RL_\beta C_\alpha$ circuit, including fractional reactive elements: inductance and capacitance and their internal series resistances. Relations for the equivalent impedance as well as resonance frequencies have been derived. It depends on four parameters: the inductance L, the capacitance C and fractional parameters α and β. Analysis of the formula describing the resonance frequencies shows that it exists only for specific values of the coefficients α and β. Further analysis showed that the phase resonance frequencies of series and parallel $RL_\beta C_\alpha$ are not identical. Magnitude resonance frequencies for the magnitude resonance of current and voltage are the same. In specific cases, the magnitude and phase resonance frequencies are identical too. For α and $\beta \longrightarrow 1$ formulas reduce to those of a classic series resonance RLC circuit.

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***Authors**: prof. dr hab. inż. Janusz Walczak, Politechnika Śląska, Wydział Elektryczny, Instytut Elektrotechniki i Informatyki Zakład Elektrotechniki Teoretycznej, Informatyki i Telekomunikacji, ul. Akademicka 2A, 44-100 Gliwice, E-mail: Janusz.Walczak@polsl.pl; mgr inż. Agnieszka Jakubowska, Politechnika Śląska, Wydział Elektryczny, Instytut Elektrotechniki i Informatyki, Zakład Elektrotechniki Teoretycznej, Informatyki i Telekomunikacji, ul. Akademicka 2A, 44-100 Gliwice, E-mail: Agnieszka.Jakubowska@polsl.pl.*