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Another Approach to the Fractional Order Derivatives.

Abstract. *This paper presents custom approach to the fractional order derivatives. It is proposed other opportunity to determine equation for fractional order derivatives calculation. All of the considerations are supported by the numerical examples that show the usefulness of certain properties of the fractional derivatives. In this article is made an attempt to answer the questions: What is the fractional derivative? When, where and why the fractional derivative should be used? Which one equation is appropriate for the fractional order derivative calculus?*

Streszczenie. *W artykule przedstawiono niestandardowe podejście do pochodnych niecałkowitego rzędu. Zaproponowana została możliwość definiowania dowolnych równań pozwalających na wyznaczenie pochodnych niecałkowitego rzędu. Teoria opisana w pracy została poparta przykładami, które pokazują użyteczność niektórych właściwości pochodnej niecałkowitego rzędu zdefiniowanej dowolnym równaniem. W artykule zrealizowana została próba odpowiedzi na pytania Czym jest pochodna niecałkowitego rzędu? Kiedy, gdzie i dlaczego pochodna niecałkowitego rzędu powinna być stosowana? Które równania do wyznaczania pochodnej niecałkowitego rzędu są poprawne?**(Inne podejście do pochodnych niecałkowitego rzędu.)***

Keywords: linear systems, fractional derivative, reduction rank of the system
Słowa kluczowe: układy liniowe, pochodna niecałkowitego rzędu, redukcja rzędu układu

Introduction

The derivative is the change of one quantity with respect to another one. Changes the value of a function $x(t)$ depending on the changes of value of the time t is called the first order time derivative, and it is denoted by $\dot{x}(t) = \frac{dx(t)}{dt}$. First order derivative can be calculated using the equation $\frac{dx(t)}{dt} = lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$. Changes of the value of the first order derivative may also occurs over time. That allows to make a second order derivative $\frac{d}{dt}\frac{dx(t)}{dt} = \frac{d^2x(t)}{dt^2}$ of the function $x(t)$. Then you can determine third order derivative $\frac{d}{dt}\frac{d^2}{dt^2}x(t) = \frac{d^3}{dt^3}x(t)$, fourth order derivative $\frac{d}{dt}\frac{d^3}{dt^3}x(t) = \frac{d^4}{dt^4}x(t)$, and so on $\frac{d}{dt}\frac{d^{k-1}}{dt^{k-1}}x(t) = \frac{d^k}{dt^k}x(t)$. It can be created a function $f(k) = \frac{d^kx(t)}{dt^k}$ where its argument k is an integer number. Whether is it possible to create the function that describes relationship between derivative and its order as of value from the fields of real number? In 1695 Marquis de L'Hopital asked Gottfried Wilhelm Laibnitz the analogous question and received a positive answer: It is possible. It can be created the fractional derivative where its order is the value from the field of real number.

Today fractional derivatives may be obtained by the use one of few known equations [1], [3]. Example of the fractional integral-derivative can be counted using the Rieman-Liouville equation:

$$(1)\quad {}_{0}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau$$

where $n-1 \leq \alpha \leq n, n \in N$.

The Grünvald-Letnikov equation:

$$(2)\quad {}_{t_0}^{GL}D_t^{\alpha} f(t) = lim_{h \to 0} \left[\frac{1}{h^{\alpha}} \sum_{i=0}^{k} a_i^{\alpha} f(t-hi) \right] 1$$

where $t - t_0 = kh$ and

$$a_1^v = \begin{cases} 1 & for \quad i = 0 \\ (-1)^i \frac{v(v-1)(v-2)\dots(v-j+1)}{i!} & for \quad i = 1, 2, \dots \end{cases}$$

The Caputo equation:

$$(3)\quad {}_{0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}}{(t-\tau)^{\alpha+1-n}} d\tau$$

where $n-1 < \alpha < n, n \in N$.

All of these equations for fractional derivatives calculation (1)-(3) are some kind of the interpolation methods of the integer order derivatives. Which one of them is correct? Which one of them, when and where should be used? What does it mean the derivative is the fractional order one?

There is no explanation of the physics of the systems with fractional derivative described by the equations ((1)-(3). All of the considerations are based on the analysis of the response of the fractional order systems [2], [1], [8], [5].

Well known fractional derivatives defined by Rieman-Liouville (1), Grünvald-Letnikov (2) or Caputo (3) have valuable properties in the world of differential equations. Unfortunately, properties of the fractional derivative are not so clear in the real systems. Knowledge of the sense of the fractional derivative is indispensable for modeling real systems. We cannot create a model of a real system, when we don't understand meaning of the derivative. For example, modeling of the ultracapacitor by the use of the fractional derivatives is not perfect. Quite often, there is only an approximation of the phenomena taking place here. In [4] is shown that the fractional derivatives used for modeling the preservation of ultracapacitor are not better from the standard derivatives (integer order). Fractional order derivative sometimes brings description of the ultracapacitor better then the standard derivative [6], [7], but unfortunately not always.

Not all of uses of the fractional derivative are explainable. Most unexplainable is modeling banking systems using fractional derivative [10] [11]. Simulation results for banking systems modeled using the fractional derivative are complex and hard to explain. It is not surprising that we can't explain the sense of the simulation results, when we don't understand sense of the fractional derivative.

Fractional order derivative depends on the history of input signal. The response of each system also depends on the history of input signal. The dependence of the derivative of the history of input signal implies a benefit in systems with delays [1]. Fractional order derivative is also used for modeling systems with infinite size of the state vector. An example might be a long line of RC [1]. Why fractional order derivatives defined by the equations (1)-(3) are not used for modeling all long lines?

Main thesis of this article is that we can determine infinite number of the methods, ways and techniques of the interpolation and calculation the fractional derivative. The equation for calculation of the fractional order derivative is arbitrary and obtained by the transformation of the main description of the system. In this article is made an attempt to answer the following questions. What is it fractional derivative? How should the fractional derivatives be calculated? When, where and why the fractional derivatives should be applied?

Reduction of the rank of the system

”The operators of fractional derivatives play a role unique filters that emit only those components that are localized on the fractal sets the process under investigation” [8]. Fractional derivatives hide some part of dynamics of the system. Using these allows us to reduce size of the state vector of the system. Supposing that the derivative can be calculated using any equation, then the rank of the system can be reduced. Finally, rank of the system can be reduced if equation for the fractional derivative calculus contains some part of dynamics of the system. Fractional derivatives defined by Rieman-Liouville (1), Grünvald-Letnikov (2) or Caputo (3) not always are useful for reduction the rank of the system.

Let’s consider fractional derivatives defined by any equation. The fractional derivative needs to represent a dynamics of the system. Using the equation corresponding to the characteristic polynomial of the system the size of the state vector can be reduced to the rank of the matrix C.

Example 1 *Let’s consider system described by the equations*

$$\begin{aligned} \dot{x}_1 &= -6x_1 - 2x_2 \\ \dot{x}_2 &= 4x_1 + u \\ y &= x_1 \end{aligned} \tag{4}$$

The characteristic polynomial of the system (4) is equal

$$det\,[Is - A] = s^2 + 6s + 8 = (s+2)(s+4) \tag{5}$$

so the fractional derivative will be defined by the equation (6).

$$x' = \ddot{x} + 6\dot{x} \tag{6}$$

Using the fractional derivative defined by the equation (6) and the system (4) there can be obtained new system described by the equations

$$\begin{aligned} {x_1}' &= -8x_1 - 2u \\ y &= x_1 \end{aligned} \tag{7}$$

The state x and the output y vectors of the system (4) for the initial conditions $x(0) = 0$ and $u(0) = 0$ are equal to [9]

$$\begin{aligned} x_1 &= \int_0^t \left(-e^{-2(t-\tau)} + e^{-4(t-\tau)}\right) u(\tau)d\tau \\ x_2 &= \int_0^t \left(2e^{-2(t-\tau)} - e^{-4(t-\tau)}\right) u(\tau)d\tau \\ y &= \int_0^t \left(-e^{-2(t-\tau)} + e^{-4(t-\tau)}\right) u(\tau)d\tau \end{aligned} \tag{8}$$

The state x and the output y vectors of the system (7) for the initial conditions $x(0) = 0$ and $u(0) = 0$ are equal to

$$y = x_1 = \int_0^t \left(-e^{-2(t-\tau)} + e^{-4(t-\tau)}\right) u(\tau)d\tau \tag{9}$$

There is no difference between the response signals of the systems (4) and (7).

Example 2 *Let’s consider system described by the equations*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ -4 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x \end{aligned} \tag{10}$$

The characteristic polynomial of the system (10) is equal

$$det[Is - A] = s^3 - s^2 - 4s + 4 = (s-1)(s-2)(s+2) \tag{11}$$

then the fractional derivative will be defined by the equation

$$x' = \dddot{x} - \ddot{x} - 4\dot{x} \tag{12}$$

Using the fractional derivative defined by the equation (12) and the system (10) there can be obtain new system described by the equations

$$\begin{aligned} {x_1}' &= -4x_1 + v \\ y &= x_1 \end{aligned} \tag{13}$$

where new input signal v is equal to $v = \ddot{u} - u$.

The state x and the output y vectors of the system (10) for the initial conditions $x(0) = 0$ and $u(0) = 0$ are equal to

(14)

$$\begin{aligned} x_1 &= \int_0^t \left(\tfrac{3}{4}e^{2(t-\tau)} + \tfrac{1}{4}e^{-2(t-\tau)}\right) u(\tau)d\tau \\ x_2 &= \int_0^t \left(\tfrac{8}{3}e^{(t-\tau)} - \tfrac{3}{2}e^{2(t-\tau)} - \tfrac{1}{6}e^{-2(t-\tau)}\right) u(\tau)d\tau \\ x_3 &= \int_0^t \left(\tfrac{8}{3}e^{(t-\tau)} - 3e^{2(t-\tau)} + \tfrac{1}{3}e^{-2(t-\tau)}\right) u(\tau)d\tau \\ y &= \int_0^t \left(\tfrac{3}{4}e^{2(t-\tau)} + \tfrac{1}{4}e^{-2(t-\tau)}\right) u(\tau)d\tau \end{aligned}$$

The state x and the output y vectors of the system (13) for the initial conditions $x(0) = 0$ and $y(0) = 0$ are equal to

$$y = x_1 = \int_0^t \left(\tfrac{3}{4}e^{2(t-\tau)} + \tfrac{1}{4}e^{-2(t-\tau)}\right) u(\tau)d\tau \tag{15}$$

There is no difference between the response signals of the systems (10) and (13).

The results of the Example 1 and the Example 2 are similar. The rank of the system can be reduced by the use of the fractional derivative defined by the equation corresponding to the characteristic polynomial.

Fractional derivatives in the real systems

This section consists considerations about physical sense of the fractional systems. Considerations presented below are based on three examples of the standard systems and theirs transform to the fractional systems. The fractional derivatives in these systems are determined from the equations that are obtained after the corresponding math transformations.

Example 3 *Let’s consider the model of the RC system where capacitor is modeled as an ideal capacitor in parallel with an ideal resistance Fig.1. Circuit presented on the Fig:1 can be*

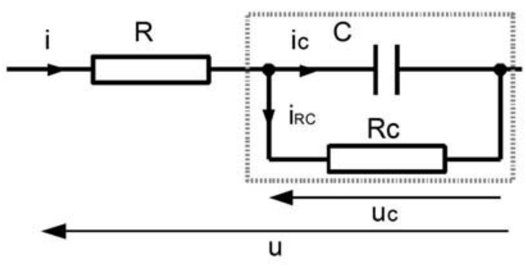

Fig. 1. The RC circuit with the model of the capacitor as of the parallel RC circuit.

described by the following equation.

$${u_C}' = \frac{-1}{RC}u_C + \frac{1}{RC}u \tag{16}$$

where ${u_C}'$ is the fractional derivative defined by the equation

$$u'_C = \frac{du_C}{dt} + \frac{1}{R_C C}u_C \tag{17}$$

The equation for fractional derivative calculation can be obtained by the use of the equation of sum of the currents in the *junction*

$$i = i_C + i_{R_C} = C\frac{du_C}{dt} + \frac{u_C}{R_C} \tag{18}$$

Using $i = Cu'_C$ we obtain the equation (17).

Example 4 *Let's consider the model of the RC system where capacitor is modeled as an ideal capacitor in series with an ideal resistor Fig.2. Circuit presented on the Fig:2 can be*

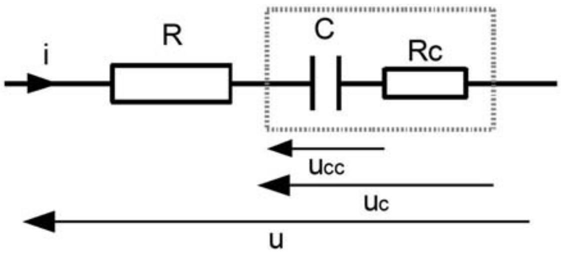

Fig. 2. The RC circuit with the model of the capacitor as of the serial RC circuit.

described by the following equation.

$$u_C{}' = \frac{-1}{RC}u_C + \frac{1}{RC}u \tag{19}$$

where $u_C{}'$ is the fractional derivative defined by the equation

$$u_C{}' = \sum_{i=1}^{\infty} (-R_C C)^{i-1} \frac{d^i u_C}{dt^i} \tag{20}$$

The equation (20) for fractional derivative calculation can be obtained from the system equations (19) and the sum of voltages in this circuit (21).

$$u = u_R + u_C = Ri + u_C \tag{21}$$

where $i = Cu_C{}'$ and $u_C{}'$ is the fractional derivative of the signal u_C. Using the equations $u_{CC} = u_C - iR_C$ and $i = C\frac{du_C}{dt}$, there can be obtained the equation for the calculation of the current

(22)

$$i = C\frac{du_C}{dt} - R_C C^2 \frac{d^2 u_C}{dt^2} + R_C^2 C^3 \frac{d^3 u_C}{dt^3} - R_C^3 C^4 \frac{d^4 u_C}{dt^4} + \ldots$$

and finally there can be obtained the equation (20).

Example 5 *Analogous result can be obtained for the circuit presented on the Fig. 3. State equation for this circuit is the*

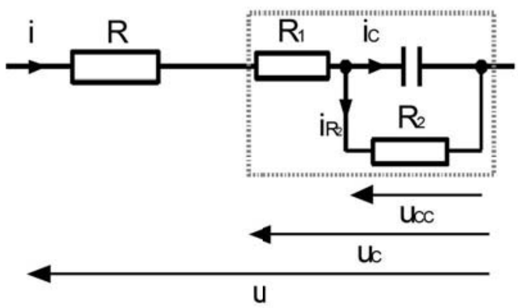

Fig. 3. The RC circuit with the model of the capacitor as of the serial R with the parallel RC circuit.

same as (16) and (19).

$$u_C{}' = \frac{-1}{RC}u_C + \frac{1}{RC}u \tag{23}$$

The difference for the previous cases is based only on change of the equation for the computation of the fractional derivative of the capacitor voltage.

(24)

$$u'_C = \frac{u_C}{C(R_1+R_2)} + \sum_{i=0}^{\infty} (-1)^i \frac{R_1^i R_2^{i+2} C^i}{(R_1+R_2)^{i+2}} \frac{d^{i+1}}{dt^{i+1}} u_C$$

If the resistance R_1 is equal to zero then the equation (24) is the same as the equation (17). If the resistance R_2 is equal to infinity then the equation (24) is the same as the equation (20).

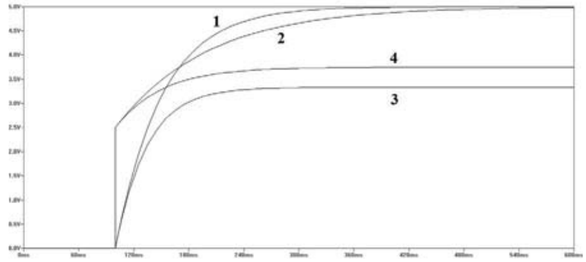

Fig. 4. Simulation results $u_C = f(t)$ for RC circuit with the model of the capacitor as of 1.the ideal capacitor 2.the serial RC circuit 3.the parallel RC circuit 4.parallel R with serial RC circuit.

The figure 4 presents voltages of the capacitors in function of the time for serial RC circuit and different models of the capacitor. The input signal is a unit increments. Graph 1 on the figure 4 shows the voltage of an ideal capacitor. Graphs 2, 3 and 4 show the voltages of the capacitor for the examples 3, 4 and 5, respectively. How easily can see the voltage of the capacitor for the examples 3 - 5 is not compatible with the voltage obtained for the ideal capacitor. We can say that the response of non ideal capacitor corresponds to the derivative of a different order then the first one.

Conclusions

In this article was taken an attempt to explain the sense of the fractional order derivatives. Meaning of the fractional order derivatives defined by the use of any equation is quite simple: The fractional order derivative contains part of the system dynamics. Not trivial problem is to clarify the physical meaning of the fractional order derivatives defined by the equations designed by Rieman-Liouville (1), Grünvald-Letnikov (2) or Caputo (3). Most probably, these derivatives consist a part of complex system dynamics, which one was not or can not be decomposed. Is it correct when they do not stem from mathematical transformations?

In the other words, if the equation for the fractional order derivative calculation contains the part of the system dynamics then the rank of this system can be reduced. In the examples 3 - 5 the rank of the system was reduced using the equation for calculation of the fractional order derivative, that contains description of part of the model of the capacitor.

Description of the system is greatly simplified, when the fractional order derivative represents a fragment of the transmittance (Example1 and Example2). Fractional order derivative is a kind of filter, which reduces the size of the state vector to its respective function and examines issues with respect to changes of the value of this function. An example of reduction of two-dimensional system to the corresponding one-dimensional system is the RC long line. Standard model of the RC long line is two-dimensional system. Using fractional derivative we obtain one dimensional system. We can say that the fractional derivative reduces size of the two-dimensional system to one dimension.

In this article, problem of the fractional order derivative was analyzed for one dimensional linear systems. All of this considerations can be extended for two dimensional systems and also nonlinear. It is also possible to design nonlinear equation for fractional order derivative calculation.

Unrestricted definition of the fractional order derivatives raises many new questions and doubts. They will be the subject of further works on the proposed approach to the problem of the fractional order derivative. An example of a real and significant issue for the proposed fractional derivative is the problem of the motion control of the truck. Dynamics of this car depends on the transported cargo. Driver steers the truck properly, even when dynamics of this car changes very much. New question arises: How to control an object to obtain response signal which is independent on the internal and unobservable dynamic?

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