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Scattering matrices of multiport radio devices

***Abstract**. This paper deals with the mathematics models and properties of three types the scattering matrices for arbitrary multiport devices normalized to separate resistive loads, separate complex loads and complex n-port load. The introduced scattering matrices describe a matching problem in different multiport structures, for example a connection of given antenna array with multiport beamformer. These mathematics models may be used for analysis of signal distribution and their disturbances in different multiport structures and for the design of any optimum multiport devices.*

***Streszczenie.** Artykuł przedstawia modele matematyczne i właściwości trzech typów macierzy rozproszenia dla dowolnych układów wielowrotnikowych normalizowanych do osobnych rezystywnych i zespolonych obciążeń oraz do obciążenia wielowrotnikowego. Wprowadzone macierze opisują problem dopasowania różnorodnych wielowrotnikowych struktur (**Macierze rozproszenia wielowrotnikowych układów radiowych**).*

Keywords: scattering matrices, multiport devices, new scattering matrix.
Słowa kluczowe: macierze rozproszenia, układy wielowrotnikowe, nowa macierz rozproszenia.

Introduction

One of the classic problems in antenna theory is a design of an optimum equalizer to match an arbitrary multiport load (antenna array) to generators [3,5]. The solution of this problem may be carried out by use of different scattering matrices normalized to diagonal or multiport networks [1-9].

The paper presents mathematical models and properties of three type the scattering matrices for given multiport devices (Fig.1): normalized to separate resistive loads [1], separate complex loads [2,3] and complex *n*-port load. The basis of the scattering matrix normalized to *n*-port complex load network is described in [4]. Using of this matrix for multiport broadband matching problem for the arbitrary antenna array is presented in [7-9]. These scattering parameters are based on *eigenvalues* and *eigenvectors* of the multiport matrices

As shown there is *infinite number* of the scattering matrices for given *n*-port complex load (or value of coupled generators) and all of these describe *the same* power distribution in given multiport networks.

The scattering matrix normalized to diagonal impedance matrix

The scattering matrix connects vectors of incident a and reflected b waves of multiport network at all: b = S a. There are different determinations of the incident and reflected waves and forms of the scattering matrix normalization according to types of loads of the multiport network (Fig.1):

- normalization to *resistive loads* (Fig.1,a) [1,3]:

$$\begin{cases} 2\sqrt{R_i}\,a_i = U_i + R_i I_i\,, \\ 2\sqrt{R_i}\,b_i = U_i - R_i I_i\,; \end{cases} \tag{1}$$

then scattering matrix $\mathbf{S}$ connected to impedance matrix $\mathbf{Z}$:

$$\mathbf{S} = \mathbf{R}^{-0.5}(\mathbf{Z}-\mathbf{R})\,(\mathbf{Z}+\mathbf{R})^{-1}\,\mathbf{R}^{0.5}\,, \tag{2}$$

where $\mathrm{R} = \{R_i\}$ - diagonal matrix of generator resisters;

- normalization to *complex loads* (Fig.1,b) [2,3,5]:

$$\begin{cases} 2\sqrt{r_i}\,a_i = U_i + z_i I_i\,, \\ 2\sqrt{r_i}\,b_i = U_i - z_i^* I_i\,, \end{cases} \tag{3}$$

where $r_i = \mathrm{Re}\, z_i$ - real parts of complex impedance's z_i; then scattering matrix $\mathbf{S}$ connected to impedance matrix $\mathbf{Z}$:

$$\mathbf{S} = \mathbf{R}_g^{-0.5}(\mathbf{Z}-\mathbf{Z}_g^*)\,(\mathbf{Z}+\mathbf{Z}_g)^{-1}\,\mathbf{R}_g^{0.5}\,, \tag{4}$$

where $\mathbf{R}_g = \mathrm{Re}\,\mathbf{Z}_g = \mathrm{Re}\{z_i\}$ – real diagonal matrix.

In the both cases a total average power absorbed by network N for arbitrary incident vector a of *separate* generators (Fig.1,a,b) is [3]:

$$P_N = \mathrm{Re}(\mathbf{U}^+\mathbf{I}) = \mathbf{a}^+(\mathbf{1}-\mathbf{S}^+\mathbf{S})\,\mathbf{a}\,; \tag{5}$$

superscript (+) denotes the *complex conjugate transpose* (also called the *hermit conjugate*) matrix [5].

A new scattering matrix normalized to *n*-port complex impedance matrix

A solution of the broadband multiport antenna matching problem for given antenna pattern may be carried out with base of design algorithm with use of ordinary scattering matrix (Fig.1,c). [1-3,5,6]:

1. Determination of amplitude-phase antenna distribution for given antenna pattern. Synthesis of the corresponding broad band divider.
2. Analysis of partial impedance's for obtained distribution.
3. Synthesis of optimal matching networks for determined partial impedance's.
4. Then it is changed:
   a) partial impedance's,
   b) amplitude-phase antenna distribution,
   c) antenna pattern (main lobe and side lobe).
5. Optimization of the broadband divider.

This is iteration process and solution of "matching-pattern" problem may be very difficulty.

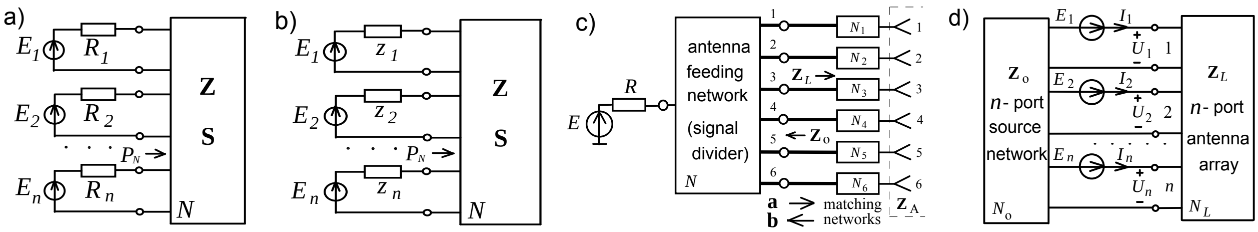

Fig.1. Structures of the multiport networks

Effective solution of this problem may be carried out by use of new multiport scattering matrices normalized to *n*-port network [4,7-9].

A new scattering matrix is constructed for given load impedance matrix Z_L and *non-diagonal* output impedance matrix Z_o of the antenna-feeding network N_o (beamformer) and corresponding equivalent structure (Fig.1,c,d).

With use equations $E = U + Z_o I$ and $U = Z_L I$, where E, U, I – corresponding vectors we have for total average power P absorbed by two-port N (Fig.1,d):

(6) $$P_N = \mathrm{Re}(\mathbf{U}^+\mathbf{I}) = \mathbf{I}^+\mathbf{R}_L\,\mathbf{I}\,,$$

where: $R_L = 0.5(Z_L + Z_L^+)$ - real part of load impedance matrix Z_L, $I = (Z_o + Z_L)^{-1}E$ - current vector.

For ideal matching case we have condition $Z^+_o = Z_L$, then incident current is $I_{in} = R_o^{-1}E$, where $R_o = 0.5(Z_o+Z_o^+)$ - real part of generator impedance matrix Z_o (Fig.1,d); then maximal average power $P_{\max}$ absorbed by two-port N is

(7) $$P_{\max} = 0.25\,\mathbf{E}^+\mathbf{R}_o^{-1}\mathbf{E}\,.$$

Then for arbitrary load network N (Fig.1,d) reflected average power P_{ref} is:

(8) $$\begin{aligned} P_{ref} &= P_{\max} - P_N = \\ &= 0.5\,\mathbf{I}^+[(\mathbf{Z}_o+\mathbf{Z}_L)^+ R_o^{-1}(\mathbf{Z}_o+\mathbf{Z}_L)-(\mathbf{Z}_L+\mathbf{Z}_L^+)]\,\mathbf{I}\,. \end{aligned}$$

After transformation we have:

(9) $$P_{ref} = 0.25\,\mathbf{I}^+(\mathbf{Z}_L^+ - \mathbf{Z}_o)\,R_o^{-1}(\mathbf{Z}_L - Z_o^+)\,\mathbf{I}\,.$$

Therefore, total average power absorbed by multiport network N is given by:

(10) $$\begin{aligned} P_N &= P_{\max} - P_{ref} = \\ &= 0.25\,\mathbf{E}^+\mathbf{R}_o^{-1}\mathbf{E} - 0.25\,\mathbf{I}^+(\mathbf{Z}_L^+ - \mathbf{Z}_o)\,\mathbf{R}_o^{-1}(\mathbf{Z}_L - \mathbf{Z}_o^+)\,\mathbf{I}. \end{aligned}$$

Further, real part of output impedance matrix $\mathbf{Z}_o$ has right- and left-standard factorizations [4,7]:

(11) $$\mathbf{R}_o = 0.5\,(\mathbf{Z}_o + \mathbf{Z}_o^+) = \mathbf{H}\mathbf{H}^+ = \mathbf{Q}^+\mathbf{Q}\,,$$

where $\mathbf{H}$, $\mathbf{Q}$, $\mathbf{H}^+$ and $\mathbf{Q}^+$- complex factors and hermit conjugation of the matrices accordingly, as in (5) superscript (+) denotes *hermit conjugate* matrix. For the reciprocal network $\mathbf{R}_o = \mathbf{R}_{o\,t}$, therefore $\mathbf{H} = \mathbf{Q}_t$, and further only *right* -standard factorization will be used. The real symmetrical positive determined matrix $\mathbf{R}_o$ has an orthogonal expansion [5]:

(12) $$\mathbf{R}_o = \mathbf{V}\left\{r_{oi}\right\}\mathbf{V}_t\,,$$

where $\mathbf{V} = \mathbf{V}^*$, $\mathbf{V}\mathbf{V}_t = \mathbf{1}$ - real orthogonal eigenvector matrix and $r_{o\,i} > 0$ - real eigenvalues of the matrix $\mathbf{R}_o$ Then factor matrix $\mathbf{H}$ has a following expansion with use of complex arbitrary unitary matrix $\mathbf{W}$, $\mathbf{W}\mathbf{W}^+ = \mathbf{1}$:

(13) $$\mathbf{H} = \mathbf{V}\left\{\sqrt{r_{o\,i}}\right\}\mathbf{W}\,.$$

In this case complex incident $\mathbf{a}$ and reflected $\mathbf{b}$ waves and complex normalized scattering matrix $\mathbf{S}$ of the multiport network (Fig.1,d) normalized to the impedance matrix $\mathbf{Z}_o$ of n - port complex source network N_o are given by:

(14) $$\begin{cases} 2\,\mathbf{H}\,\mathbf{a} = \mathbf{U} + \mathbf{Z}_o\mathbf{I} = (\mathbf{Z}_L + \mathbf{Z}_o)\,\mathbf{I} = \mathbf{E}, \\ 2\,\mathbf{H}^+\mathbf{b} = \mathbf{U} - \mathbf{Z}_o^+\mathbf{I} = (\mathbf{Z}_L - \mathbf{Z}_o^+)\,\mathbf{I}, \end{cases} \quad \mathbf{b} = \mathbf{S}\,\mathbf{a}$$

If to insert these formulae for incident $\mathbf{a}$ and reflected $\mathbf{b}$ waves in (10) we have known equation for total average power absorbed by multiport network:

(15) $$P_N = P_{\max} - P_{ref} = \mathbf{a}^+\mathbf{a} - \mathbf{b}^+\mathbf{b}\,.$$

After comparison of equations (10-15) we make an *important conclusion*: there are an *infinite number* of the incident a and reflected b waves and multiport scattering matrices S for given multiport complex load through the arbitrary unitary matrix W. But all of these scattering matrices determine *the same* power dissipation in multiport structure (Fig.1,d) because a hermitian form is used for power calculation (10) and, as a result of further transformation of formulas, both matrices W and W^+ disappear.

The scattering matrix $\mathbf{S}$ may be expressed by means of the impedance matrices:

(16) $$\mathbf{S} = (\,\mathbf{H}^+)^{-1}(\mathbf{Z}_L - \mathbf{Z}_o^+)\,(\,\mathbf{Z}_L + \mathbf{Z}_o)^{-1}\,\mathbf{H}\,,$$

where $\mathbf{Z}_L$ and $\mathbf{Z}_o$ - impedance matrix of the multiport load N_L and source networks N_o (Fig.1,d).

It is possible to express $\mathbf{S}$ in terms of $\mathbf{Y}_\Sigma$ for the "augmented" n - port:

(17) $$\mathbf{S} = \mathbf{C} - 2\,\mathbf{H}_t\mathbf{Y}_\Sigma\mathbf{H}\,,$$

where: $\mathbf{C} = (\mathbf{H}^*)^{-1}\,\mathbf{H} = \mathbf{W}_t\mathbf{W}\,,\quad \mathbf{C}\,\mathbf{C}^+ = \mathbf{1}$

is the complex unitary matrix determined by arbitrary unitary matrix $\mathbf{W}$ from (13);

(18) $$\mathbf{Y}_\Sigma = \left(\mathbf{Z}_L + \mathbf{Z}_o\right)^{-1}$$

is an admittance matrix of the "augmented" *n* - port (Fig.1,d).

Thus the complex normalized scattering matrix is

(19) $$\mathbf{S} = \mathbf{W}_t\left[\mathbf{1} - 2\left\{\sqrt{r_{oi}}\right\}\mathbf{V}_t\,\mathbf{Y}_\Sigma\,\mathbf{V}\left\{\sqrt{r_{oi}}\right\}\right]\mathbf{W}\,.$$

The elements of $\mathbf{S}$ are determined by means of:

(20) $$b_j = \sum_{i=1}^{n} s_{ij}\;a_i\,,$$

where a_i are the elements of incident wave vector:

(21) $$\mathbf{a} = 0.5\,\mathbf{H}^{-1}\,\mathbf{E}\,.$$

The determination of s_{ij} is may be made from the condition of *one nonzero* element from vector a and *all nonzero* elements from E:

(22) $$\begin{aligned} & a_i = E_i/(2\,h_{ii})\,,\quad a_j = 0\,,\quad i \neq j\,, \\ & E_j = (h_{ji}\;E_i)/h_{ii}\,,\quad i\,,\,j = 1,2,\ldots n\,, \end{aligned}$$

where h_{ii}, h_{ij} are the elements of factor matrix $\mathbf{H}$ (13).
For determined scattering matrix and arbitrary excitation vector $\mathbf{a}$ voltage and current vectors are:

(23) $$\begin{aligned} \mathbf{U} &= [\,\mathbf{Z}_o^+(\mathbf{H}^+)^{-1} + \mathbf{Z}_o\mathbf{H}_t^{-1}\mathbf{S}]\,\mathbf{a} \\ \mathbf{I} &= [\,(\mathbf{H}^+)^{-1} + \mathbf{H}_t^{-1}\mathbf{S}]\,\mathbf{a}. \end{aligned}$$

It is may be proven then total average power absorbed by whole multiport N_L is given by

(24) $$P_N = \mathrm{Re}(\mathbf{U}^+\mathbf{I}) = \mathbf{a}^+\mathbf{a} - \mathbf{b}^+\mathbf{b} = \mathbf{a}^+(\mathbf{1} - \mathbf{S}^+\mathbf{S})\mathbf{a} = \mathbf{a}^+\mathbf{D}\,\mathbf{a}\,,$$

where matrix $D = 1 - S^+S$ named by *dissipation* matrix.

Then normalized total average power absorbed by network N_L (*Rayleigh ratio*) is limited by the minimum and the maximum eigenvalues of the dissipation matrix D:

$$(25)\qquad \begin{aligned} &d_{\min} \le P_N / P_{\max} = (\mathbf{a}^+ \mathbf{D}\, \mathbf{a}) / \mathbf{a}^+\mathbf{a} \le d_{\max} \\ &\mathbf{D} = \mathbf{V}\left\{ d_i \right\} \mathbf{V}^+ , \quad d_i = d_i^* , \quad \mathbf{V}\mathbf{V}^+ = \mathbf{1} \end{aligned}$$

and what's more the Rayleigh ratio is arrived eigenvalues $d_{\min}$ i $d_{\max}$ when the incident wave vectors a are parallel to the corresponding eigenvectors of the matrix D. In this case the optimization and the matching problem of the arbitrary multiport network reduce to the maximization of the minimum eigenvalues of the dissipation matrix D at the given frequency band.

A new scattering matrix for cascade connection of multiports

Consider the cascade connection of multiport networks: an excitation of n - port load N_β from set of separate generators by multiport coupling network N (Fig.2) [7,8].

Determine the complex incident and reflected waves for the scattering matrix $\mathbf{S}$ of the multiport coupling network normalized to separate complex internal source impedance's $z_{\alpha i}$ from ports "α" and to n - port load network impedance $\mathbf{z}_{o\beta}$ from ports "β". Then entire normalization impedance matrix corresponding to source network N_o (Fig.1,d) is

$$(26)\qquad \mathbf{Z}_o = \begin{bmatrix} \mathbf{z}_{o\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{z}_{o\beta} \end{bmatrix} , \quad \mathbf{z}_{o\alpha} = \left\{ z_{\alpha i} \right\}, \quad i = 1,2,\ldots n ,$$

for which the first n generators are uncoupled, but the rest of these are zero (Fig.2). In this case the network structure in Fig.2 may be considered as the particular network of Fig.1,d.

Then for ports "α" the complex normalized incident and reflected waves are:

$$(27)\qquad \begin{cases} 2\mathbf{H}_{o\alpha}\mathbf{a}_\alpha = \mathbf{U}_\alpha + \mathbf{z}_{o\alpha}\mathbf{I}_\alpha , \\ 2\mathbf{H}^+_{o\alpha}\mathbf{b}_\alpha = \mathbf{U}_\alpha - \mathbf{z}^+_{o\alpha}\mathbf{I}_\alpha , \end{cases}$$

where $\mathbf{U}_\alpha$ and $\mathbf{I}_\alpha$ are voltage and current vectors of multiport network N for these ports (Fig.2); $\mathbf{H}_{o\alpha}$ and $\mathbf{H}^+_{o\alpha}$ are diagonal matrices - factors of the real part of diagonal matrix $\mathbf{R}_{o\alpha}$ of uncoupled generators:

$$(28)\qquad \mathbf{R}_{o\alpha} = (\mathbf{z}_{o\alpha} + \mathbf{z}^*_{o\alpha})/2 = \left\{ r_{\alpha i} \right\} = \mathbf{H}_{o\alpha}\mathbf{H}^*_{o\alpha} ,$$

where: $\mathbf{H}_{o\alpha} = \mathbf{H}^*_{o\alpha} = \left\{ h_{\alpha i} \right\} = \left\{ \sqrt{r_{\alpha i}} \right\}$.

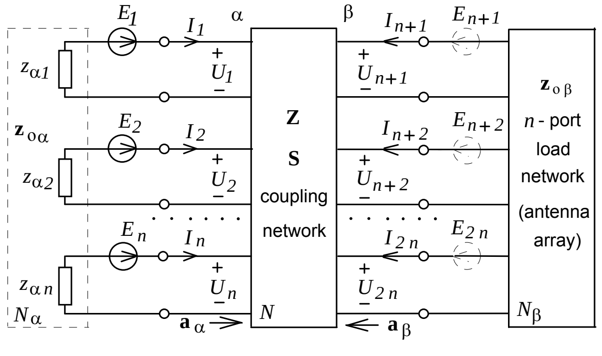

Fig.2. The structure of cascade connection of multiport networks

Consequently incident and reflected waves for these ports may be determined by the same as (2) [1,3]:

$$(29)\qquad \begin{cases} 2\sqrt{r_{\alpha i}}\, a_{\alpha i} = U_i + z_{\alpha i}\, I_i , \\ 2\sqrt{r_{\alpha i}}\, b_{\alpha i} = U_i - z^*_{\alpha i}\, I_i , \end{cases} \quad i = 1, 2, \ldots n ,$$

where $r_{\alpha i} = \mathrm{Re}\, z_{\alpha i}$. Then incident waves exactly correspond to components of the vector $\mathbf{E}_\alpha$:

$$(30)\qquad a_{\alpha i} = E_{\alpha i} / 2\sqrt{r_{\alpha i}} , \quad i = 1, 2, \ldots n .$$

For ports "β" the incident $\mathbf{a}$ and reflected $\mathbf{b}$ wave vectors of the multiport network N normalized to the total matrix $\mathbf{z}_{o\beta}$ of n - port complex load network N_β are given by general relations (9) [4,7-9]:

$$(31)\qquad \begin{cases} 2\mathbf{H}_{o\beta}\, \mathbf{a}_\beta = \mathbf{U}_\beta + \mathbf{z}_{o\beta}\, \mathbf{I}_\beta , \\ 2\mathbf{H}^*_{o\beta}\, \mathbf{b}_\beta = \mathbf{U}_\beta - \mathbf{z}^*_{o\beta}\, \mathbf{I}_\beta , \end{cases}$$

where $\mathbf{H}_{o\beta}$- complex matrix factor of the right-standard factorizations of real part of $\mathbf{z}_{o\beta}$ as (11):

$$(32)\qquad \mathbf{R}_{o\beta} = (\mathbf{z}_{o\beta} + \mathbf{z}^+_{o\beta})/2 = \mathbf{H}_{o\beta}\mathbf{H}^+_{o\beta} .$$

Then every component of incident wave vector $\mathbf{a}_\beta$ for ports "β" is determined by entire vector $\mathbf{E}_\beta = [E_{n+1}, \ldots E_{2n}]_t$:

$$(33)\qquad \mathbf{a}_\beta = 0.5\, \mathbf{H}^{-1/2}_{o\beta}\mathbf{E}_\beta ;$$

then a condition of *one nonzero* element of $\mathbf{a}_\beta$ is *all nonzero* elements from $\mathbf{E}_\beta$ in total case:

(34)

$$a_{\beta i} = E_{\beta i} / 2 h_{\beta ii}, a_{\beta j} = 0, \;\; i \ne j, \;\; E_{\beta j} = E_{\beta i} h_{\beta ji} / h_{\beta ii} , \\ i, j = n+1, n+2, \ldots 2n ,$$

where $h_{\beta ii}$, $h_{\beta ji}$ are elements of factor matrix $\mathbf{H}_{o\beta}$ (27).

It is known that the symmetrical real matrix $\mathbf{R}_{o\beta}$ has an orthogonal expansion as in (12) [5]:

$$(35)\qquad \mathbf{R}_{o\beta} = \mathbf{V}\left\{ r_{\beta i} \right\}\mathbf{V}_t ,$$

where $\mathbf{V} = \mathbf{V}^*$, $\mathbf{V}\mathbf{V}_t = \mathbf{1}$ - real orthogonal eigenvector matrix and $r_{\beta i} = r^*_{\beta i} > 0$ - real positive eigenvalues of the matrix $\mathbf{R}_{o\beta}$.

Therefore factor matrix $\mathbf{H}_{o\beta}$ has total following expansion:

$$(36)\qquad \mathbf{H}_{o\beta} = \mathbf{V}\left\{ \sqrt{r_{\beta i}} \right\}\mathbf{W} ,$$

where $\mathbf{W}$ - is an arbitrary complex unitary matrix, $\mathbf{W}\mathbf{W}^* = \mathbf{1}$.

Consequently there is an infinite number of the matrices $\mathbf{H}_{o\beta}$ and vectors $\mathbf{a}_\beta$ and $\mathbf{b}_\beta$ (27), (29) connected to arbitrary matrix $\mathbf{W}$ for given n-port complex load $\mathbf{z}_{o\beta}$ (Fig.2).

The complex normalized scattering matrix $\mathbf{S}$ of the network N (Fig.2) is provided by the block relations:

$$(37)\qquad \mathbf{b} = \mathbf{S}\mathbf{a} , \quad \begin{bmatrix} \mathbf{b}_\alpha \\ \mathbf{b}_\beta \end{bmatrix} = \begin{bmatrix} \mathbf{S}_{\alpha\alpha} & \mathbf{S}_{\alpha\beta} \\ \mathbf{S}_{\beta\alpha} & \mathbf{S}_{\beta\beta} \end{bmatrix} \begin{bmatrix} \mathbf{a}_\alpha \\ \mathbf{a}_\beta \end{bmatrix}$$

and for reciprocal networks:

$$(38)\qquad \mathbf{S}_{\alpha\alpha} = \mathbf{S}_{\alpha\alpha t} , \; \mathbf{S}_{\beta\beta} = \mathbf{S}_{\beta\beta t} , \; \mathbf{S}_{\alpha\beta} = \mathbf{S}_{\beta\alpha t} .$$

It is a matter of direct verification to prove the correlation between the block $\mathbf{S}_{\alpha\alpha}$ of complex normalized scattering matrix $\mathbf{S}$ and impedance matrices from ports "α" [2,3]:

(39) $$\mathbf{S}_{\alpha\alpha} = \left\{\sqrt{r_{\alpha i}}\right\}^{-1}\left[\mathbf{Z}_{\alpha} - \left\{z^{*}_{\alpha i}\right\}\right]\left[\mathbf{Z}_{\alpha} + \left\{z_{\alpha i}\right\}\right]^{-1}\left\{\sqrt{r_{\alpha i}}\right\},$$

where $\mathbf{Z}_{\alpha}$ - impedance matrix of the n - port network consist of $2n$ - port coupling network N and n - port load network N_{β} (Fig.2); $z_{\alpha i}$ - internal source impedance's.

The entire scattering matrix $\mathbf{S}$ may be expressed by means of the admittance matrix [7]:

(40) $$\mathbf{S} = \begin{bmatrix} \mathbf{C}_{o\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{o\beta} \end{bmatrix} - 2\begin{bmatrix} \mathbf{H}_{o\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{o\beta t} \end{bmatrix}\mathbf{Y}_{\Sigma}\begin{bmatrix} \mathbf{H}_{o\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{o\beta} \end{bmatrix},$$

where: $\mathbf{Y}_{\Sigma}$ - admittance matrix of "augmented" $2n$ - port N:

(41) $$\mathbf{Y}_{\Sigma} = \begin{bmatrix} \mathbf{Y}_{\alpha\alpha} & \mathbf{Y}_{\alpha\beta} \\ \mathbf{Y}_{\beta\alpha} & \mathbf{Y}_{\beta\beta} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{\alpha\alpha} + \left\{z_{\alpha i}\right\} & \mathbf{Z}_{\alpha\beta} \\ \mathbf{Z}_{\beta\alpha} & \mathbf{Z}_{\beta\beta} + \mathbf{z}_{o\beta} \end{bmatrix}^{-1},$$

$\mathbf{Z}_{\alpha\alpha}$, $\mathbf{Z}_{\alpha\beta}$, $\mathbf{Z}_{\beta\alpha}$, $\mathbf{Z}_{\beta\beta}$ - blocks of impedance matrix of $2n$ - port N:

(42) $$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_{\alpha\alpha} & \mathbf{Z}_{\alpha\beta} \\ \mathbf{Z}_{\beta\alpha} & \mathbf{Z}_{\beta\beta} \end{bmatrix},$$

$\mathbf{C}_{o\alpha}$ and $\mathbf{C}_{o\beta}$ are all-pass ("phase") matrices determined by normalization matrices $\mathbf{H}_{o\alpha}$ and $\mathbf{H}_{o\beta}$:

(43) $$\mathbf{C}_{o\alpha} = \left(\mathbf{H}^{*}_{o\alpha}\right)^{-1}\mathbf{H}_{o\alpha} = \left\{\sqrt{r_{\alpha i}}\right\}^{-1}\left\{\sqrt{r_{\alpha i}}\right\} = \mathbf{1},$$

(44) $$\mathbf{C}_{o\beta} = \left(\mathbf{H}^{*}_{o\beta}\right)^{-1}\mathbf{H}_{o\beta} = \mathbf{W}_{t}\mathbf{W},\ \mathbf{C}_{o\beta} = \mathbf{C}_{o\beta t} = \left(\mathbf{C}^{*}_{o\beta}\right)^{-1}.$$

According to (28) phase matrix $\mathbf{C}_{o\alpha}$ is equal to unit on imaginary axis (43); but $\mathbf{C}_{o\beta}$ is whole symmetrical unitary matrix determined by the complex arbitrary unitary matrix $\mathbf{W}$ (36), (44).

Then the blocks of the scattering matrix $\mathbf{S}$ may be expressed by means of blocks (41) of the admittance matrix $\mathbf{Y}_{\Sigma}$ of the "augmented" $2n$ - port (Fig.2):

(45) $$\begin{cases} \mathbf{S}_{\alpha\alpha} = \mathbf{1} - 2\,\mathbf{R}^{1/2}_{o\alpha}\,\mathbf{Y}_{\alpha\alpha}\,\mathbf{R}^{1/2}_{o\alpha}, \\ \mathbf{S}_{\alpha\beta} = -2\,\mathbf{R}^{1/2}_{o\alpha}\,\mathbf{Y}_{\alpha\beta}\,\mathbf{H}_{o\beta}, \\ \mathbf{S}_{\beta\beta} = \mathbf{C}_{o\beta} - 2\,\mathbf{H}_{o\beta t}\,\mathbf{Y}_{\beta\beta}\,\mathbf{H}_{o\beta}. \end{cases}$$

The elements of block $\mathbf{S}_{\alpha\alpha}$ with (39), (44) may be expressed by the following parameters:

(46) $$s_{\alpha ii} = \frac{Z_{\alpha i} - z^{*}_{\alpha i}}{Z_{\alpha i} + z_{\alpha i}} = 1 - 2y_{\alpha ii}\,r_{\alpha i},\ s_{\alpha ji} = -2\,y_{\alpha ji}\sqrt{r_{\alpha i}\,r_{\alpha j}},$$

where $Z_{\alpha i}$ represent the impedance seen looking into port i of side"α" under matched terminations by impedance's $z_{\alpha i}$ for $E_{\alpha j} = 0,\ j \neq i,\ i, j = 1, 2, \ldots n$ and load network $\mathbf{z}_{o\beta}$ (Fig.2); $y_{\alpha ii}$, $y_{\alpha ji}$ are elements of block $\mathbf{Y}_{\alpha\alpha}$ of the admittance matrix $\mathbf{Y}_{\Sigma}$. For block $\mathbf{S}_{\beta\beta}$ used (36) and (44) we obtain:

(47) $$\mathbf{S}_{\beta\beta} = \mathbf{W}_{t}\left[\mathbf{1} - 2\left\{\sqrt{r_{\beta i}}\right\}\mathbf{V}_{t}\,\mathbf{Y}_{\beta\beta}\,\mathbf{V}\left\{\sqrt{r_{\beta i}}\right\}\right]\mathbf{W}.$$

The expressions (41), (42) and (43) are shown that block $\mathrm{S}_{\alpha\alpha}$ unequivocally determined by internal source impedance's $z_{\alpha i}$; but there is an infinite number of the scattering matrix blocks $\mathrm{S}_{\alpha\beta}$, $\mathrm{S}_{\beta\beta}$ and then entire matrix S for given n - port complex load through the arbitrary unitary matrix W. All of these scattering matrices determine the same power dissipation of the network (Fig.2).

Power parameters of new scattering matrix

Consider an excitation of the cascade connection of n-ports from side "α" only: there are arbitrary vector $\mathbf{E}_{\alpha}$ and $\mathbf{E}_{\beta} = \mathbf{0}$ (Fig.2).In this case voltage and current vectors are:

(48) $$\begin{cases} \mathbf{U}_{a} = \mathbf{R}^{-0.5}_{o\alpha}\,(\mathbf{z}^{*}_{o\alpha} + \mathbf{z}_{o\alpha}\mathbf{S}_{\alpha\alpha})\,\mathbf{a}_{\alpha} \\ \mathbf{I}_{a} = \mathbf{R}^{-0.5}_{o\alpha}\,(\mathbf{1} - \mathbf{S}_{\alpha\alpha})\,\mathbf{a}_{\alpha} \end{cases},\ \begin{cases} \mathbf{U}_{\beta} = \mathbf{z}_{o\beta}\,\mathbf{H}^{-1}_{o\beta t}\,\mathbf{S}_{\beta\alpha}\,\mathbf{a}_{\alpha} \\ \mathbf{I}_{\beta} = -\,\mathbf{H}^{-1}_{o\beta t}\,\mathbf{S}_{\beta\alpha}\,\mathbf{a}_{\alpha} \end{cases}.$$

In this case the total average power P_{α} absorbed by n - port load network N_{β} and coupling network N for the given excitation vector $\mathbf{E}_{\alpha}$ and $\mathbf{E}_{\beta} = \mathbf{0}$ (Fig.2) is given by

(49) $$P_{\alpha} = \mathbf{a}^{+}_{\alpha}\mathbf{a}_{\alpha} - \mathbf{b}^{+}_{\alpha}\mathbf{b}_{\alpha} = \mathbf{a}^{+}_{\alpha}\left(\mathbf{1} - \mathbf{S}^{+}_{\alpha\alpha}\mathbf{S}_{\alpha\alpha}\right)\mathbf{a}_{\alpha} = \mathbf{a}^{+}_{\alpha}\mathbf{D}_{\alpha\alpha}\mathbf{a}_{\alpha},$$

where: $\mathbf{a}_{\alpha}$ is determined by $\mathbf{E}_{\alpha}$ (26); matrix $\mathbf{D}_{\alpha\alpha} = \mathbf{1}\text{-}\mathbf{S}^{+}_{\alpha\alpha}\mathbf{S}_{\alpha\alpha}$ named by *dissipation* matrix [5,7-9]. It is the *hermetian* matrix $\mathbf{D}_{\alpha\alpha} = \mathbf{D}^{+}_{\alpha\alpha}$ and the unitary similar to the diagonal positive real matrix of the eigenvalues $d_{\alpha i}$ [7,8]:

(50) $$\mathbf{D}_{\alpha\alpha} = \mathbf{V}_{\alpha}\left\{d_{\alpha i}\right\}\mathbf{V}^{+}_{\alpha},\quad d_{\alpha i} = d^{*}_{\alpha i},\quad \mathbf{V}_{\alpha}\mathbf{V}^{+}_{\alpha} = \mathbf{1},$$

where $\mathbf{V}_{\alpha}$ is the complex unitary matrix (called the modal) of the eigenvectors of matrix $\mathrm{D}_{\alpha\alpha}$.

For *ideal matching* a maximum total average power absorbed by whole network is:

(51) $$\mathbf{S}_{\alpha\alpha} = \mathbf{0},\ \mathbf{D}_{\alpha\alpha} = \mathbf{1},\ P_{\max} = \mathbf{a}^{+}_{\alpha}\mathbf{a}_{\alpha}.$$

Then normalized average power absorbed by N and N_{β} for the arbitrary vector $\mathbf{a}_{\alpha}$ is the *normalized hermetian form* of dissipation matrix $\mathrm{D}_{\alpha\alpha}$ (*Rayleigh ratio*) [5,7-9]:

(52) $$\frac{P_{\alpha}}{P_{\max}} = \frac{\mathbf{a}^{+}_{\alpha}\left(\mathbf{1} - \mathbf{S}^{+}_{\alpha\alpha}\mathbf{S}_{\alpha\alpha}\right)\mathbf{a}_{\alpha}}{\mathbf{a}^{+}_{\alpha}\mathbf{a}_{\alpha}} = \frac{\mathbf{a}^{+}_{\alpha}\mathbf{D}_{\alpha\alpha}\,\mathbf{a}_{\alpha}}{\mathbf{a}^{+}_{\alpha}\mathbf{a}_{\alpha}}.$$

The value of the Rayleigh ratio is a function of the incident vector $\mathbf{a}_{\alpha}$. It is shown, that the Rayleigh ratio has stationary quantities equal to the eigenvalues of the corresponding hermitian matrix; this means that the normalized total average power $P_{\alpha}/P_{\max}$ is limited by the **minimum** and the maximum eigenvalues of the dissipation matrix $\mathbf{D}_{\alpha\alpha}$:

(53) $$d_{\alpha\min} \le \frac{P_{\alpha}}{P_{\max}} = \frac{\mathbf{a}^{+}_{\alpha}\mathbf{D}_{\alpha\alpha}\mathbf{a}_{\alpha}}{\mathbf{a}^{+}_{\alpha}\mathbf{a}_{\alpha}} \le d_{\alpha\max},$$

and what's more the Rayleigh ratio is arrived eigenvalues $d_{\alpha\min}$ i $d_{\alpha\max}$ when the incident wave vectors $\mathbf{a}_{\alpha}$ are parallel to the corresponding eigenvectors $\mathbf{V}_{\alpha}$ of the matrix $\mathbf{D}_{\alpha\alpha}$ (50).

As forms (49) - (52), the normalized average power $P_{\beta}/P_{\max}$ absorbed by load network N_{β} for the arbitrary vector $\mathbf{a}_{\alpha}$ is the *normalized hermetian form* of hermitian matrix $\mathrm{S}^{+}_{\beta\alpha}\mathrm{S}_{\alpha\beta}$ (*Rayleigh ratio*) and limited by the **minimum** and the **maximum** eigenvalues of this hermitian matrix:

(54) $$d_{\beta\min} \le \frac{P_{\beta}}{P_{\max}} = \frac{\mathbf{a}^{+}_{\alpha}\mathbf{S}^{+}_{\beta\alpha}\mathbf{S}_{\beta\alpha}\mathbf{a}_{\alpha}}{\mathbf{a}^{+}_{\alpha}\mathbf{a}_{\alpha}} \le d_{\beta\max}.$$

Consequently the optimization and the matching problem of the arbitrary multiport network is reduced to two tasks:

- a maximization of the minimum eigenvalues $d_{\alpha\min}$ of the dissipation matrix $\mathrm{D}_{\alpha\alpha}$ (50) at the given frequency band for the optimization of the total average power P_{α} absorbed by n - port load network N_{β} and coupling network N

for *arbitrary* excitation vector E_α (the total average source power);

- a maximization of the minimum eigenvalues $d_{\beta\,\min}$ of the hermitian matrix $S^{+}_{\beta\alpha}S_{\alpha\beta}$ at the given frequency band for the optimization of the total average power P_β absorbed by load network N_β only for *arbitrary* excitation vector E_α.

If $2n$ - port coupling network N (Fig.2) is lossless, the values of the power are equal ($P_\alpha = P_\beta$), the unitary condition for scattering matrix **S** is provided:

$$\mathbf{D}_{\alpha\alpha} = \mathbf{1} - \mathbf{S}^{+}_{\alpha\alpha}\mathbf{S}_{\alpha\alpha} = \mathbf{S}^{+}_{\beta\alpha}\mathbf{S}_{\alpha\beta} \tag{55}$$

and these two tasks are reduced to any one.

Conclusions

The considered mathematical models and the structures of three types the scattering matrices for arbitrary multiport devices normalized to separate resistive and complex loads and complex *n*-port load may by use for the evaluation of the broadband problems of arbitrary multiport structures. It is presented structures and properties of a new scattering matrix for general case and for multiport cascade connection.

The calculation of the blocks of the new scattering matrices and power parameters based on the eigenvalues and eigenvectors of the multiport network matrices may be carried out by use the standard computer packets. As shown there is *infinite number* of these scattering matrices for given *n*-port complex load (or value of coupling generators) and all of these describe *the same* power distribution in given multiport networks.

The introduced scattering matrices describe a matching problem in different multiport structures, for example a connection of given antenna array with multiport beamformer. These mathematical models and parameters may be used for the design of an optimum multiport reactive equalizer to match an arbitrary passive multiport load to coupled generators and the optimization of multiport antenna-matching network of the different structures. These mathematics models may be used for analysis of signal distribution and their disturbances in different multiport structures and for the design of any optimum multiport devices.

Solution of these tasks is very important for EMC and EMD analysis of the radio systems with complicate transmit and receive parts of whole system.

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