Andrzej PIEGAT, Karina TOMASZEWSKA

West Pomeranian University of Technology, Szczecin, Poland

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Defuzzification with Optimal Representation Method

***Abstract**. Though the number of existing defuzzification methods is considerable, scientists further work on the new methods trying to elaborate more perfect ones and eliminate imperfection and weak-points of existing methods. The paper proposes a new defuzzification method, which in the authors opinion, has chances for scientific acknowledgement because it is based on a new approach. In this method there is no aggregation of activated rule conclusions as in many other methods. Instead of aggregation, the method determines the best, optimal fuzzy representation of the activated conclusions and then finds the optimal crisp representation. The main advantage of the proposed method is simplicity of calculations.*

***Streszczenie.** Choć liczba istniejących metod defuzyfikacji jest znaczna, naukowcy prowadzą dalsze prace nad nowymi metodami, starając się opracować bardziej doskonałe i wyeliminować niedoskonałości istniejących już metod. W pracy zaproponowano nową metodę defuzyfikacji, która w opinii autorów, ma szanse zdobycia uznania w środowisku naukowym, ponieważ pokazuje zupełnie nowe podejście. W sposobie tym nie ma agregacji zaktywowanych reguł wnioskowych tak jak w wielu innych metodach. Zamiast agregacji, metoda wyznacza najlepszą, optymalną rozmytą reprezentację aktywowanych wniosków, a następnie znajduje optymalną punktową reprezentację. Główną zaletą proponowanej metody jest prostota obliczeń.(**Defuzyfikacja metodą Optymalnej Reprezentacji**).*

Keywords: defuzzification, fuzzy systems, fuzzy modeling, fuzzy control.
Słowa kluczowe: defuzyfikacja, systemy rozmyte, modelowanie rozmyte, sterowanie rozmyte.

Introduction

There exist a considerable number of defuzzification methods (D-methods) of the aggregated inference result of a rule base [1]. Out of more methods proposed so far, let us list 22 methods [2] here: adaptive integration method (AI), basic defuzzification distributions method (BADD), constraint decision defuzzification method (CDD), center of area method (COA), center of gravity method (COG), extended center of area method (ECOA), extended quality method (EQM), fuzzy clustering defuzzification method (FCD), fuzzy mean method (FM), first of maximum method (FOM), generalized level set defuzzification method (GLSD), indexed center of gravity method (ICOG), influence value method (IV), last of maximum method (LOM), mean of maxima method (MeOM), middle of maximum method (MOM), quality method (MOM), quality method (QM), random choice of maximum method (RCOM), semi-linear defuzzification method (SLIDE), [3] , [4] , weighted fuzzy mean method (WFM) and Golden Ratio defuzzification method (GR). Literature on defuzzification is rich and in References only small part of publications on the subject was cited. The defuzzification component, as important part of a fuzzy model, has been investigated since about 1970 and in eyes of fuzzy community this subject seems to be completely investigated and finished. However, it is not true. Again and again scientists come back to the defuzzification and present new ideas on how to realize this operation. Examples can be papers [5] and [6] in 2015. When we solve a problem we should try to choose the best defuzzification method. However, what is the meaning of "the best method"? It means that a multicriterion consisting of a series of component criteria specific for the problem and depending on the scientist individual preferences should be used. E.g. in [7] the defuzzification algorithm has been hardware-realized and oriented and hence it should not require a great memory for operations, see also [8]. Author of [9] gives advices how to choose a D-method to a given application. There exist also interesting works presenting evaluations of particular D-methods, e.g. [10] and [9]. In [10] authors draw the readers' attention to the fact that optimality of a D-method depends on specificity of the concrete application. E.g. methods using "maxima" are more appropriate for fuzzy reasoning systems while methods using "area" are more suitable for fuzzy controllers. The numbers of possible criteria and of required properties used for D-methods evaluation is considerable. Examples of desired properties (criteria) are: property of a scale invariance, of monotony, satisfying the triangular conorm criteria, property of x-translation, of x-scalling, of continuity, of computational efficiency, and of transparency (of easy understanding). Thus the choice of the optimal D-method is multicriterial one, scientist-dependent (significance evaluation of particular component-criteria dependent and application-dependent). But generally we can evaluate quality of particular D-methods on the basis of their popularity (use frequency) in the fuzzy community. Many publications on D-methods inform that 2 methods are mostly used: Center of Gravity (COG) method and Mean of Maxima (MeOM) method. The second method is derived from COG-one and calculates the mean of all elements of the core of a fuzzy set. Hence, in fact it can be called "the COG of the core" method. COG-method calculates, according to [10] the expected value when fuzzy set *A* is considered [3] to be a probability distribution (1).

$$(1)\quad COG(A)=\sum_{x_{\min}}^{x_{\max}} A(x)\cdot x \Big/ \sum_{x_{\min}}^{x_{\max}} A(x)$$

Mean of Maxima (MeOM) calculates the mean of all elements of a fuzzy set *A* (2).

$$(2)\quad MeOM(A)=\sum\nolimits_{x\in cor(A)} x/|cor(A)|$$

The Optimal Representation (OpR) method proposed in this paper is based on a new (according to authors' knowledge) approach to defuzzification.

General description of the Optimal Representation (OpR) method

Let us assume that the rule base describing the dependence $y=f(x_1,\ldots,x_k)$ between input variables $\{x_1,\ldots, x_k\}$ and the output variable y of a system consists of n rules (3) in which i is the rule number $i=1\div n$ and j is the variable number $j=1\div k$.

(3) $IF\ (x_1\ is\ B_{11})\ AND\ (x_2\ is\ B_{12})\ AND \ldots AND\ (x_k\ is\ B_{1k})\ THEN\ (y\ is\ A_1)$

$\vdots$

$IF\ (x_1\ is\ B_{n1})\ AND\ (x_2\ is\ B_{n2})\ AND \ldots AND\ (x_k\ is\ B_{nk})\ THEN\ (y\ is\ A_n)$

Each of the particular rules R_i can be interpreted as an expert that possess knowledge about value of the output variable y only in a local sub-domain of the system domain $X_{1i}\times X_{2i}\times,\ldots,\times X_{ki}$, where X_{ji} are domains of particular input variables. If on the system inputs a vector $\{x_1,x_2, \ldots,x_k\}$ of concrete numerical variable values is applied then as the inference result conclusions of particular rules R_i will be activated (fired) to degrees w_i. Example of rule-conclusions activation is shown in Fig. 1.

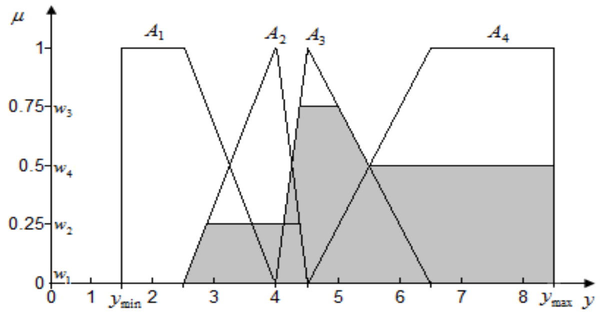

Fig.1. Example of rule conclusion activation as a result of inference in a fuzzy system with non-extended border fuzzy sets.

In the case of only the maximal activation (firing) of the border sets $A_1(y)$ and $A_n(y)$, after defuzzification border values $y_{\min}$ or $y_{\max}$ should be calculated. In a fuzzy model with non-extended border sets and with COG defuzzification, achievement of $y_{\min}$ or $y_{\max}$ is not possible. To enable it the output variable domain $[y_{min},y_{max}]$ should artificially be extended to $[y_{min} - S_{A1}, y_{max} + S_{AN}] = [y_{min}^*, y_{max}^*]$, where S_{A1} and S_{AN} are supports of border fuzzy sets. In the case of the example from Fig.1 the artificially extended domain of variable y is shown in Fig.2.

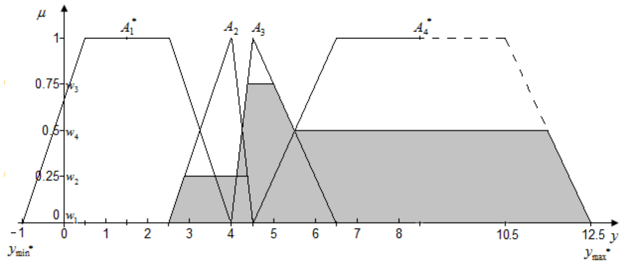

Fig.2.Example fuzzy sets of the output variable *y* activated by Center of Gravity method.

Depending on concrete values of input variables $\{x_1,x_2,\ldots,x_k\}$ the inference process causes different activation $\{w_{A1},\ldots,w_{An}\}$ of particular fuzzy sets A_i of rule conclusions. The activation degrees are frequently interpreted as the conclusions' truth. In the mostly used COG method, on their basis, the aggregated MF of the output variable y is then determined. For activated MFs from Fig.2 it has form shown in Fig.3.

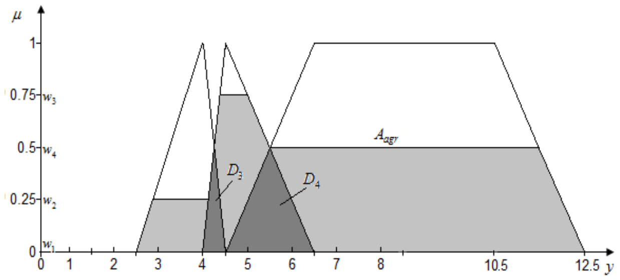

Fig.3. Conclusions of a rule base with repeating fragments D_3 and D_4 of fuzzy sets A_2, A_3 and A_4^* (extended A_4) aggregated with operator Max.

One can observe that fragments D_3 and D_4 of fuzzy sets A_2, A_3 and A_4^* (extended A_4) occur two times in conclusions (in the pair A_2, A_3 and in A_3, A_4). It causes certain inaccuracy in calculations of the crisp result y^* of defuzzification. With COG method in the example shown in Fig.3 the crisp result is achieved y_{COG}^*=7.539.

In the OpR defuzzification method following steps should be made:

1. Determine generalizing membership function (GMF) for all fuzzy sets of the output variable y.
2. Using activation degrees w_{Ai} of particular conclusions A_i determine the optimal representative fuzzy set A_i of all activated conclusions. To this aim choose the optimality criterion.
3. For the optimal representative set A_r determine the optimal crisp (singleton) representation value y_R. To this aim choose the optimality criterion.

Further on the OpR defuzzification method will be shown on an example.

Example of defuzzification with Optimal Representation method

Let us assume that in the course of inference fuzzy sets of the output variable y has been activated as shown in Fig.2 and values of the activations are as follows: $w_{A1}^* = 0$, $w_{A2} = 0.25$, $w_{A3} = 0.75$, $w_{A4}^* = 0.50$. For calculative convenience activation values will be normalized according to (4).

$$(4)\quad w_{AiN} = w_{Ai} / \sum_{i=1}^{4} w_{Ai}, i = 1 \div 4$$

After the normalization the activation values are as follows: $w_{A1N}^* = 0$, $w_{A2N} = 1/6$, $w_{A3N} = 3/6$, $w_{A4N}^* = 2/6$. In the OpR-method conclusions A_i of particular rules are understood as fuzzy expert-evaluations informing about approximate value of the output variable y corresponding to the input vector $\{x_1, \ldots, x_k\}$ applied to the system. Each of rules R_i is an expert possessing knowledge about approximate value of variable y in its local sub-domain of the full input-domain $X_1 \times X_2 \times \ldots \times X_k$. Activation values of conclusions do not necessarily have to be interpreted in the traditional way as in the COG-method, as shown in Fig.2 and Fig.3. They can be interpreted as competence degrees of particular experts (rules) to determine output value y for given input vector $\{x_1, x_2, \ldots, x_k\}$. Hence activation values can graphically be interpreted as greyness-degrees of particular conclusions A_i, see Fig.4. One can also observe that fuzzy sets A_i are of different qualitative type: triangle- and trapezoid-ones.

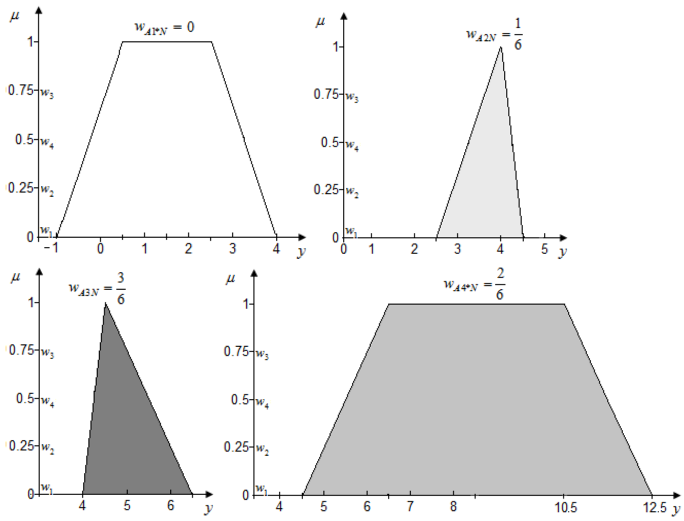

Fig.4. Conclusion activations w_{Ai} as competence degrees of rules R_i influencing full membership functions of conclusions (greyness degrees) used for determining of the resulting crisp output value y_R of a fuzzy model.

Step 1 of the OpR method – determining the generalizing membership function (GMF) for all fuzzy sets A_i of the output variable y.
Among MFs of variable y occur 2 qualitative types of MFs: triangle and trapezium functions. The triangle MF is a special case of trapezoid MF, Fig.5.

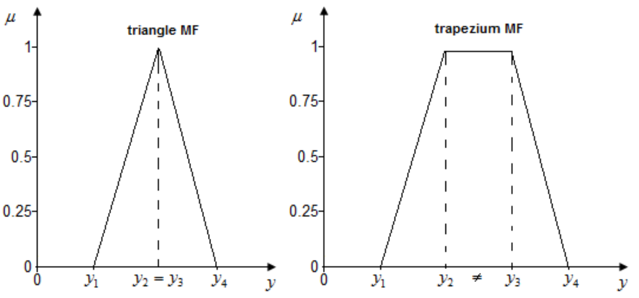

Fig.5. The triangle MF as a special case of the trapezium MF, y_r- characteristic points determining the trapezium MF, r=1÷4.

As the generalizing MF such MF should be understood which contains (or from which can be derived as a special case) all types of MFs occurring in rules conclusions of the output variable y. Generalizing MFs can be determined not only for linear-segment MFs but also for other function types: Gauss MFs, polynomial MFs [11], etc. However, it is not subject of this paper. Trapezium MF is given by (5)

$$\text{(5)}\quad \begin{aligned} & y < y_1 : \mu = 0 \\ & y_1 \le y < y_2 : \mu = (y - y_1)/(y_2 - y_1) \\ & y_2 \le y < y_3 : \mu = 1 \\ & y_3 \le y \le y_4 : \mu = (y_4 - y)/(y_4 - y_3) \\ & y > y_4 : \mu = 0 \end{aligned}$$

As can be seen from (5) the trapezium MF is fully determined by positions y_1, y_4 of its support and positions y_2, y_3 of its core. Hence, these points can be called characteristic points of the trapezium MF.

Step 2 of the OpR-method – Determining the MF optimally representing all weighted rule conclusions.
In Step 2 a fuzzy set A_R is to be determined that optimally, according to the assumed criterion, represents all activated conclusions A_i with taking into account normalized weight (activation) coefficients w_{AiN}. As the optimality criterion various criteria can be assumed depending on the modeler's preferences and the system specificity. Frequently chosen and used criterion is the criterion S of the minimal sum of square, weighted differences of the representing set A_R and the activated conclusion sets A_i. Let us denote by $y_{1R},\dots,y_{4R}$ characteristic points of the representing MF$_R$, where r=1÷4 is the number of characteristic point. The criterion S –value should be minimized (6).

$$\text{(6)}\quad S = \sum_{i=1}^{4} w_{AiN} (y_{rR} - y_r)^2$$

The optimal value of the representing characteristic function-point y_{rR} of the MF$_R$ satisfies the condition (7).

$$\text{(7)}\quad \frac{\partial S}{\partial y_{rR}} = 0$$

From condition (7) condition (8) can be derived.

$$\text{(8)}\quad y_{rR} \sum_{i=1}^{4} w_{AiN} - \sum_{i=1}^{4} w_{AiN} y_{ri} = 0$$

Because the sum satisfies condition $\sum_{i=1}^{4} w_{AiN} = 1$ then the optimal value of y_{rR} has the form of (9)

$$\text{(9)}\quad y_{rR} = \sum_{i=1}^{4} w_{AiN} y_{Ri,} i = 1 \div 4$$

Thus, the optimal value of y_{rR} is the weighted average. For particular parameters of MF$_R$ following values are achieved:

$$y_{1R} = 0 \cdot (-1) + \frac{1}{6} \cdot 2.5 + \frac{3}{6} \cdot 4 + \frac{2}{6} \cdot 4.5 = 3.9167$$

$$y_{2R} = 5.0833$$

$$y_{3R} = 5.75$$

$$y_{4R} = 8.1667$$

Fig.6b. shows the optimal MF$_R$ representing 3 activated ($w_{AiN} > 0$) rule conclusions A_i shown on Fig.6a.

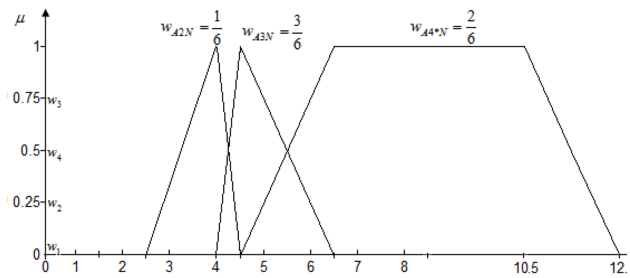

Fig.6a.Activated rule conclusions A_i

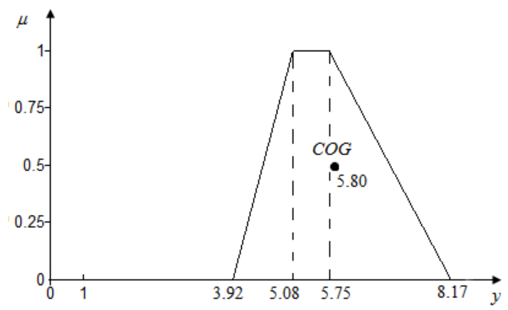

Fig.6b. The optimal MF$_R$ of the set A_R

Fig.6. Activated to various degrees rule conclusions A_2, A_3, A_{4*} and a membership function MF$_R$ of the set A_R optimally representing them. The value y_{COGR} = 5.8052 is the crisp value optimally representing the representative fuzzy set A_R.

The representative set A_R is the optimal fuzzy conclusion from the full rule base. However, we can further try to represent this fuzzy conclusion by only one crisp value y_R. To this aim also an optimality criterion has to be chosen. One of possible criteria can be the criterion of minimal sum of weighted square differences between y_R and all possible values of the output y contained in the representative fuzzy set A_R. Hence, the sum D has to be minimized in respect of y_R (10).

$$\text{(10)}\quad D = \int_{y_{1R}}^{y_{AR}} (y_R - y)^2 \cdot \mu(y) dy$$

The optimal value y_R can be found from condition (11)

$$\frac{\partial D}{\partial y_R} = 2\int_{y_{1R}}^{y_{4R}} (y_R - y)\cdot \mu(y)dy = 0 \quad (11)$$

The solution is given by (12).

$$y_R = \frac{\int_{y_{1R}}^{y_{4R}} y\cdot \mu(y)dy}{\int_{y_{1R}}^{y_{4R}} \mu(y)dy} \quad (12)$$

Thus, the optimal singleton representation y_r of the representative fuzzy set A_R is according to the assumed criterion (6) the center of gravity. Other types of criteria will result in other representations y_R. In the considered example, according to the OpR method, the singleton representation has value $y_R = 5.805$.

Conclusions

The paper presented, according to authors' knowledge, a new defuzzification method, which in the first step determines the optimal, representative fuzzy set A_R and in the second step the crisp value y_R, which in the optimal way represents the representative fuzzy set A_R. The method called shortly OpR-method possess following features:

1. It is simple in calculations, considerably easier than COG-method because the geometric form of the representative fuzzy set A_R is simpler than usually complicated form of the aggregated conclusion set achieved from the inference in the frame of COG-method. Also the support (uncertainty) of the set A_R is considerably smaller.
2. In the OpR-method the geometric form of the representative ser A_R and the crisp representative value y_R is influenced by all activated conclusions of the rule base. The influence of the geometric conclusions' forms does not exist in many defuzzification methods where fuzzy conclusions are replaced by singleton values.
3. In defuzzification with OpR-method take part full conclusions of all activated rules. In commonly used D-methods as COG-one conclusions occur only one time. In COG-method fragments of neighbor conclusions which overlap in the aggregated conclusion of the rule base take part in defuzzification also only one time. In the OpR-method they occur as frequently as the rule conclusions containing them. In this method nor full conclusions or their parts are eliminated from defuzzification.
4. OpR-method has no probabilistic character as Yager and Filev suggested in the case of COG-method in [3,4,10]. It is a method of replacing many uncertain conclusions by their simpler and less uncertain representation based on the assumed optimality criterion. Choice of this criterion is the individual matter of the fuzzy system modeler and there does not exist any "objective" criterion for this choice. Center of gravity of a fuzzy set is only one of possible crisp representations of a possibility distribution corresponding to the criterion of the minimal sum of square differences (10).
5. It seems that the OpR-method well suits both to fuzzy modeling and control problems and to fuzzy reasoning because of its "democratic" approach: its democratic taking into account all rule conclusions and their significance (competence) degrees.

***Authors**: prof. dr hab. inż. Andrzej Piegat, Zachodniopomorski Uniwersytet Technologiczny w Szczecinie, Wydział Informatyki, ul. Żołnierska 49, 71-210 Szczecin, E-mail: apiegat@wi.zut.edu.pl; mgr inż. Karina Tomaszewska, Zachodniopomorski Uniwersytet Technologiczny w Szczecinie, Wydział Informatyki, ul. Żołnierska 49, 71-210 Szczecin, E-mail: mtomaszewska@wi.zut.edu.pl.*

REFERENCES

[1] Saletic Z., Velasevic D. M. Velasevic, Mastorakis N.E., Analysis of basic defuzzification methods.
[2] Wikipedia: www.wikipedia.org – defuzzification.
[3] Yager R.R., Filev D.P., Adaptive defuzzification for fuzzy system modeling, *Proceedings of NAFIPS 1992*, (1992), Vol.1, 135-142.
[4] Yager R.R., Filev D.P., SLIDE: a simple adaptive defuzzification method, *IEEE Transactions on Fuzzy Systems*, (1993), No.1, 69-78.
[5] Anzilli L., Faccinetti G., A new proposal of defuzzyfication of intuitionistic fuzzy quantities, *Procedings of Fourteenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets*, Kraków, Poland, (2015). (to be published by Springer in 2016).
[6] Dobrosielski W.T., Szczepański J., Zarzycki H., A proposal for a metod of defuzzification based on the Golden Ratio-GR, *Proceedings of Fourteenth International Workshop on Intuitionistic Fuzzy Sets and Generalized Nets*, Kraków, Poland, (2015).
[7] Eisele M., Hentschel K., Kunegund T., Hardware realization of fast defuzzification by adaptive integration, *Proceedings of the fourth International Conference on Microelectronics for Neural Networks and Fuzzy Systems*, (1994), 318-323.
[8] Watanabe H., Detloff W.D., Yount K.E., A VLSI fuzzy logic controller with reconfigurable, cascadable architecture, *IEEE Journal of Solid-State Circuits,* No.2, 376-382.
[9] Runkler T.A., Selection of appropriate defuzzification methods using application specific properties, *IEEE Transactions on Fuzzy Systems*, 5 (1), (1997), 72-79.
[10] Leekwijck W.V., Kerre E.E., Defuzzification: criteria and classification, *Fuzzy Sets and Systems*, 108, (1999), 159-178.
[11] Piegat A., Fuzzy modeling and control, *Physica-Verlag*, A Springer-Verlag Company, (2001).