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Mathematical modelling of real transmission shafts and mechanical connections with clearances

***Abstract**. In the paper complex kinematic models of mechanical connections between the driving motor and the working mechanism are analyzed. As a result, the mathematical models of connections considered have been formulated. The scope of carried out analysis has included: long elastic transmission shaft with point-wise distributed lumped parameters, mechanical connection with clearance, single-stage elastic gear train with clearance and multipath rigid mechanical power transmission with clearances.*

***Streszczenie.** W pracy poddano analizie złożone modele kinematyczne połączeń mechanicznych silnika z mechanizmem roboczym. Sformułowano modele matematyczne rozważanych połączeń. W zakresie przeprowadzonej analizy znalazły się: długi wał napędowy z dyskretnie rozłożonymi parametrami skupionymi, połączenie mechaniczne z luzem, jednostopniowa sprężysta przekładnia mechaniczna z luzem oraz wielodrożna sztywna transmisja mocy mechanicznej z luzami (**Modelowanie matematyczne rzeczywistych wałów napędowych oraz połączeń mechanicznych z luzami**).*

Keywords: mechanical power transmissions in drive systems, electrical and mechanical analogies, kinematic and mathematical models.
Słowa kluczowe: transmisje mocy mechanicznej w napędach, analogie elektryczno-mechaniczne, modele kinematyczne i matematyczne.

Introduction

Electric motors are connected with working mechanisms via transmission shafts that are elements of mechanical power transmissions. The shafts have various lengths and cross-sections. Mechanical power transmissions include also driving machineries, i.e. gear trains and clutches. Depending on length and cross-section, transmission shafts can demonstrate different susceptibilities to the impact of torsional moment, as measured by a value of torsional angle. In the case of short mechanical connections, values of torsional angles are insignificant and they may be omitted by assumption of rigid mechanical connections. In the case of longer mechanical connections the values of torsional angles cannot be ignored and such connections should be considered as the elastic ones [1,2,3,4,5,6,7,8,9].

The mass of real transmission shaft (or any element) is distributed continuously. In simple terms, the mass of long elastic shafts may be considered as a parameter distributed continuously along the longitudinal axis of the shaft, in particular, as a parameter concentrated in a point (lumped parameter). Representing the real transmission shaft with continuous mass distribution by kinematic model based on lumped parameters causes discrepancies in results of analysis in relation to accurate models [4], but considerably simplifies this analysis. In addition, these discrepancies decrease with the number of points of concentration in the model. Representing a drive system, containing elastic elements, by the model with two points of concentration allows for simplification of the model as much as possible but it may not in all cases be applied. Such mathematical description is best suited to mechanical systems containing the connection between electric motor and moving unit of working machine via long shaft with negligible moment of inertia in contrast to the significant moments of inertia of the abovementioned elements of mechanical system. In other cases, dividing a long elastic shaft with continuous mass distribution into several shorter elements, described by lumped parameters such as mass, elasticity and damping of the i-th element, it is possible to obtain the results of computer simulation not essentially different from the results obtained if a long elastic shaft is divided into infinite number of elements, that corresponds with the wave model [9]. The process of the abovementioned division is referred to as discretization of kinematic structure.

Toothed gears and clutches causes clearances in mechanical system, in which the mechanical power may be transmitted between motor and working mechanism after so-called “taking in the clearance”, i.e. when the one part of mechanism has been turned in relation to the another part by a certain angle $2\Delta\gamma_0$.

In the paper complex kinematic models of mechanical connections between the driving motor and the working mechanism are analyzed. The equivalent circuits, typical for electrical systems, are defined for the mechanical systems concerned.

Long elastic transmission shaft with point-wise distributed lumped parameters

The kinematic model of long transmission shaft, divided into m elements as a result of discretization, is depicted in Fig. 1, whereas the corresponding equivalent circuit is shown in Fig. 2, where $J_1,\ldots,\ J_m,\ C_{s,12},\ldots,\ C_{s,m\text{-}1,m},\ S_{c,12},\ldots,\ S_{c,m\text{-}1,m},$ $D_{12},\ldots,\ D_{m\text{-}1,m}$ are moments of inertia, torsional elasticity coefficients, torsional susceptibility coefficients and mechanical friction coefficients (resistances) of the respective elements of divided transmission shaft; D_1, D_m are mechanical friction coefficients defined for bearings.

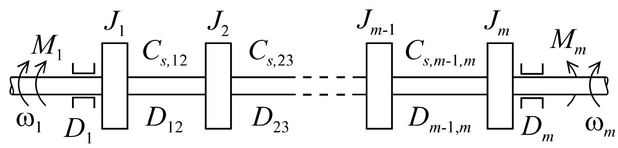

Fig. 1. Kinematic model of long elastic transmission shaft with lumped parameters

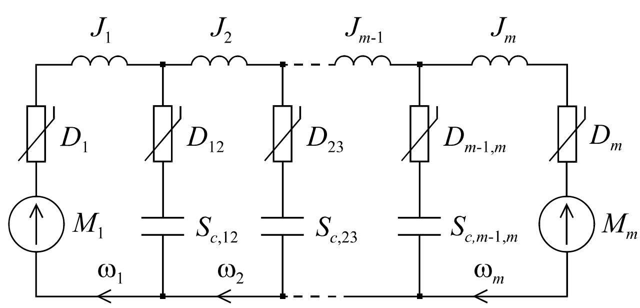

Fig. 2. Equivalent circuit of transmission shaft depicted in Fig. 1

The m equations of torques and $m-1$ equations of angular velocities may be written for the mechanical connection concerned:

$$M_1 - D_1\omega_1 - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_1\omega_1\right) - D_{12}\left(\omega_1-\omega_2\right) - M_{c,12} = 0$$

$$\left.\begin{aligned} &M_{c,k-1,k} + D_{k-1,k}\left(\omega_{k-1}-\omega_k\right) - \\ &\quad - D_{k,k+1}\left(\omega_k-\omega_{k+1}\right) - \\ &\quad - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_k\omega_k\right) - M_{c,k,k+1} = 0 \end{aligned}\right\} \quad \text{for} \quad k = 2,\ldots,m-1 \tag{1}$$

$$\begin{aligned} &M_{c,m-1,m} + D_{m-1,m}\left(\omega_{m-1}-\omega_m\right) - \\ &\quad - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_m\omega_m\right) - D_m\omega_m - M_m = 0 \end{aligned}$$

$$\omega_k - \omega_{k+1} = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(S_{c,k,k+1}M_{c,k,k+1}\right) \quad \text{for} \quad k = 1,\ldots,m-1$$

On the basis of the equivalent circuit (Fig. 2) it can be concluded that the long elastic shaft may be analyzed in a similar way as lossless transmission line if the internal friction inside the shaft has been omitted.

Mechanical connection with clearance

The analysis of a mechanism with clearances requires consideration of two operating stages of such mechanism. In the first stage, i.e. from the time at which the clearance appears to the time at which the process of so-called "taking in the clearance" is finished, two parts of mechanism, located on both sides of the element showing the clearance, are rotating independently of one another with angular velocities ω_1 and ω_2, respectively. In this stage the mechanical power is not transmitted between the abovementioned parts of the mechanism. It corresponds with the equivalent circuit depicted in Fig. 3, whereas, the kinematic model of the exemplary element showing the clearance (mechanical clutch) is depicted in Fig. 4.

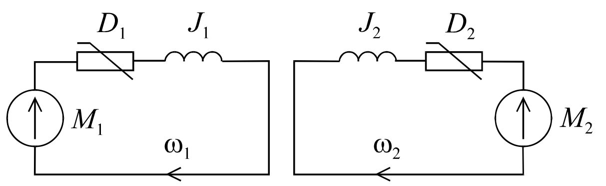

Fig. 3. Equivalent circuit of the considered mechanical connection during the process of "taking in the clearance"

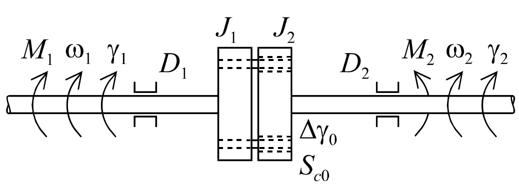

Fig. 4. Kinematic model of mechanical clutch showing the clearance

The angle $\Delta\gamma$ between both parts of mechanism may be defined for the first operating stage as follows:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\Delta\gamma\right) = \omega_1 - \omega_2 \tag{3}$$

with the initial condition $\Delta\gamma(0) = \pm\Delta\gamma_0$, i.e.:

$$\Delta\gamma = \int_0^t\left(\omega_1-\omega_2\right)\mathrm{d}\tau \pm \Delta\gamma_0 \tag{4}$$

The process of "taking in the clearance" is followed by the second operating stage of mechanism. This stage is related to the transmission of mechanical power between previously separated parts of mechanism. In the real mechanical systems the non-gradual change in velocity is not possible. It deals with the considered case, which may be interpreted as an elastic collision of two parts of mechanism, previously rotating with different velocities. In the second operating stage of mechanism (after "taking in the clearance"), the angular velocities of both parts of mechanism are no longer independent of one another. The element of mechanical connection showing the clearance is then represented by the following equivalent circuit, which, in principle, is the same as for the elastic connection, but the sources of elasticity are different in these systems:

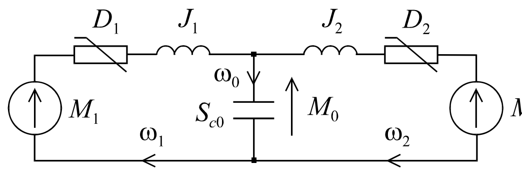

Fig. 5. Equivalent circuit of the considered mechanical connection after "taking in the clearance"

One single equivalent circuit can take into account both operating stages of mechanism with clearances (Fig. 6). Apart from the elements of the equivalent circuits for each operating stage (Figs. 3 and 5), the additional circuit, representing the clearance phenomenon, occurs in the abovementioned equivalent circuit:

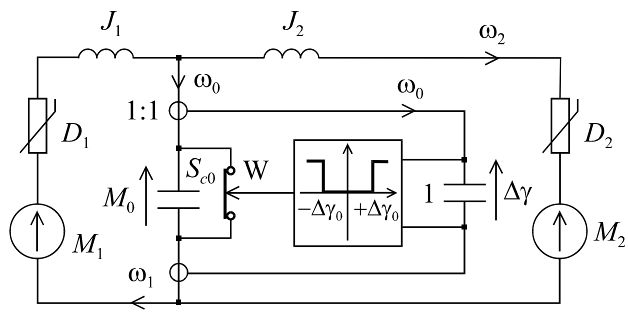

Fig. 6. Equivalent circuit of mechanical connection showing the clearance

During the time within which the breaker W is closed (the stage of "taking in the clearance"), the moment M_0, originating from the torque transmitted via mechanical connection, is equal to zero. Since the process of "taking in the clearance" is finished, the breaker W is no longer closed and the mechanical connection commences to develop the moment M_0 until the steady-state is reached. After change of rotation direction of one part of mechanism or as a result of reduction of ω_1 value relative to ω_2 value, the negative value of difference $\omega_1 - \omega_2$ causes decrease in the value of moment M_0 until it reaches the zero value ($M_0 = 0$). From this time the one part of mechanism has no longer the contact with the another and the breaker W is no longer opened due to the fact that from this time $\Delta\gamma < \Delta\gamma_0$.

It should be noted that values of parameter S_{c0}, being the mechanical susceptibility coefficient of contact between both parts of the element showing the clearance, are relatively low in contrast to the high values of analogical parameter S_c of typical elastic connections (long shafts). As a consequence, from the start of elastic collision of the abovementioned parts, the difference between velocities ω_1 and ω_2 reduces abruptly. In addition, the significant initial differences between velocities ω_1 and ω_2 are accompanied by the unsafe impact loads of mechanism elements. It corresponds with high rigidity (i.e. low susceptibility) of the collision area of both parts belonging to the element showing the clearance.

Single-stage elastic gear train with clearance

The single-stage elastic gear train with clearance is the example of a complex mechanical connection between the driving motor and the working machine. The equivalent circuit of the considered mechanical connection is depicted in Fig. 7.

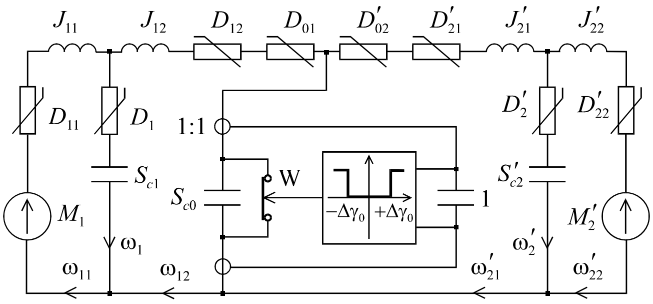

Fig. 7. Equivalent circuit of single-stage elastic gear train with clearance

The respective elements of mechanism are represented in the equivalent circuit (Fig. 7) by the following quantities and parameters: M_1, M_2 are torque of driving motor and load torque, respectively, J_{11}, J_{22} are equivalent moments of inertia of rotating lumped masses concentrated at the rotor of driving motor and moving unit of working machine, respectively, D_{11}, D_{22}, D_1, D_2 are mechanical friction coefficients corresponding with internal losses of driving motor and working machine, respectively, as well as the losses coming from bearings of transmission shafts on both sides of gear train and internal losses inside the shaft structure, S_{c1}, S_{c2} are torsional susceptibility coefficients of transmission shafts, J_{12}, D_{12}, D_{01}, S_{c0}, D_{02}, D_{21}, J_{21} are parameters of gear train, incl. equivalent moments of inertia defined for the gears, torsional susceptibility coefficient defined for the contact between the gears as well as mechanical friction coefficients defined for the bearings and the contact between the gears.

The equations of torques describing the considered mechanism after "taking in the clearance" (the second operating stage) as well as the dependencies linking torsional moments and differences in angular velocities on both sides of shafts and gear train are as follows:

$$M_1 - D_{11}\omega_{11} - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_{11}\omega_{11}\right) - D_1\omega_1 - M_{10} = 0$$

$$M_{10} + D_1\omega_1 - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_{12}\omega_{12}\right) - \left(D_{12} + D_{01}\right)\omega_{12} - M_0 = 0$$

$$M_0 - \left(D'_{02} + D'_{21}\right)\omega'_{21} - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J'_{21}\omega'_{21}\right) - D'_2\omega'_2 - M'_{20} = 0$$

(4) $$M'_{20} + D'_2\omega'_2 - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J'_{22}\omega'_{22}\right) - D'_{22}\omega'_{22} - M'_2 = 0$$

$$\omega_1 = \omega_{11} - \omega_{12} = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(S_{c1}M_{10}\right)$$

$$\omega_{12} - \omega'_{21} = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(S_{c0}M_0\right)$$

$$\omega'_2 = \omega'_{21} - \omega'_{22} = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(S'_{c2}M'_{20}\right)$$

During the period of "taking in the clearance" (the first operating stage), the mutual moment M_0 is equal to zero ($M_0 = 0$), therefore, this term and depending mechanical friction coefficients are reduced in the second equation and the third equation of the system of equations 4 as well as the penultimate equation of the system of equations 4, which links the parts of mechanism on both sides of element showing the clearance (gear train), is reduced. Thus, the equations of torques describing the considered mechanism during the process of "taking in the clearance" as well as the dependencies linking torsional moments and differences in angular velocities on both sides of shafts are as follows:

$$M_1 - D_{11}\omega_{11} - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_{11}\omega_{11}\right) - D_1\omega_1 - M_{10} = 0$$

$$M_{10} + D_1\omega_1 - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_{12}\omega_{12}\right) - D_{12}\omega_{12} = 0$$

$$D'_{21}\omega'_{21} - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J'_{21}\omega'_{21}\right) - D'_2\omega'_2 - M'_{20} = 0$$

(5) $$M'_{20} + D'_2\omega'_2 - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J'_{22}\omega'_{22}\right) - D'_{22}\omega'_{22} - M'_2 = 0$$

$$\omega_1 = \omega_{11} - \omega_{12} = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(S_{c1}M_{10}\right)$$

$$\omega'_2 = \omega'_{21} - \omega'_{22} = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(S'_{c2}M'_{20}\right)$$

where: $\omega' = \omega N$, $M' = MN^{-1}$, $D' = DN^{-2}$, $J' = JN^{-2}$ are gear train output quantities and parameters in terms of gear train input, $N = n_2 / n_1$ is gear ratio, n_1, n_2 are numbers of teeth of driver gear and follower gear, respectively.

Multipath rigid mechanical power transmission with clearances

The kinematic model of multipath mechanical power transmission is depicted in Fig. 8, whereas the equivalent circuit corresponding with the multipath mechanical power transmission with clearances is shown in Fig. 9.

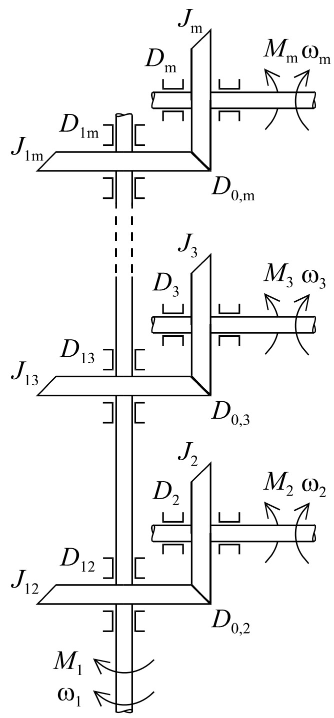

Fig. 8. Kinematic model of multipath mechanical power transmission

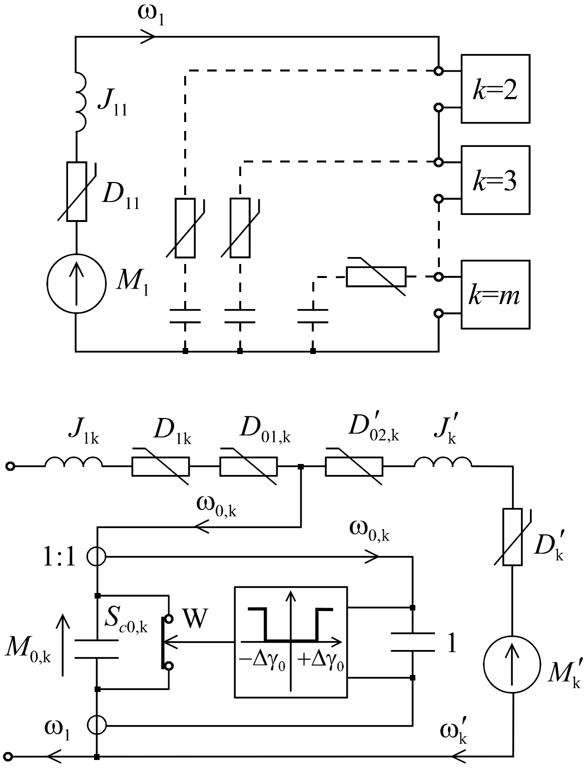

Fig. 9. Equivalent circuit of multipath mechanical power transmission with clearances

The dashed line has been used in order to show in Fig. 9 the manner of connection of torsional susceptibilities (capacities) and viscous friction resistances to the equivalent circuit in the case of consideration of elasticity dealing with the main shaft of the studied mechanical power transmission.

The considered equivalent circuit (Fig. 9) is described by the following equations:

- after "taking in the clearance" ($|\Delta\gamma_k| \geq \Delta\gamma_{0k}$)

$$M_1 - D_{11}\omega_1 - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_{11}\omega_1\right) - $$
$$- \sum_{k=2}^{m}\left(\left(D_{1k} + D_{01,k}\right)\omega_1 + \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_{1k}\omega_1\right) + M_{0,k}\right) = 0$$

$$\text{(6)} \qquad \left.\begin{aligned} &M_{0,k} - \left(D'_k + D'_{02,k}\right)\omega'_k - \\ &- \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J'_k\omega'_k\right) - M'_k = 0 \end{aligned}\right\} \quad \text{for} \quad k = 2,\dots,m$$

$$\omega_1 - \omega'_k = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\Delta\gamma_k\right) = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(S_{c0,k}M_{0,k}\right) \quad \text{for} \quad k = 2,\dots,m$$

- during the process of "taking in the clearance" ($|\Delta\gamma_k| < \Delta\gamma_{0k}$)

$$M_1 - D_{11}\omega_1 - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_{11}\omega_1\right) - \sum_{k=2}^{m}\left(D_{1k}\omega_1 + \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_{1k}\omega_1\right)\right) = 0$$

$$\text{(7)} \qquad D'_k\omega'_k + \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J'_k\omega'_k\right) + M'_k = 0 \quad \text{for} \quad k = 2,\dots,m$$

$$\omega_1 - \omega'_k = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\Delta\gamma_k\right) \quad \text{for} \quad k = 2,\dots,m$$

where: D_{11} is equivalent mechanical friction coefficient defined for driving motor, J_{11} is equivalent moment of inertia of rotating masses concentrated at the rotor of driving motor as lumped mass (parameters omitted in Fig. 8).

In the above equations the elasticity of main shaft of mechanical power transmission has not been taken into account.

Considering the multipath mechanical power transmission without clearances, the equivalent circuit may be simplified up to non-branched one. The equation of torques for the abovementioned simplified case is given as follows:

$$\text{(8)} \qquad \begin{aligned} &M_1 - D_1\omega_1 - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_1\omega_1\right) - \\ &- \sum_{k=2}^{m}\left(\left(D'_{0,k} + D'_k\right)\omega_1 + \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J'_k\omega_1\right) + M'_k\right) = 0 \end{aligned}$$

where: $D_1 = D_{11} + D_{12} + D_{13} + \dots + D_{1m}$ and $J_1 = J_{11} + J_{12} + J_{13} + \dots + J_{1m}$, $D'_{0,k} = D_{01,k} + D'_{02,k}$.

Conclusions

In the paper the complex mathematical models of mechanical connections between the driving motor and the working mechanism, including the real transmission shaft and the multipath mechanical power transmissions, are analyzed. The equivalent circuits, typical for electrical systems, are defined for the mechanical systems concerned. The equations of torques and equations of angular velocities, analogous to the Kirchhoff's lows based equations applied in mathematical analysis of branched electrical circuits, are defined.

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