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Estimation method for measurements with heavy-tailed noise variance

***Abstract**. The paper presents a new method of signal estimation in systems with measurement channel corrupted by noise which variance is random process with heavy-tailed distribution. The noise model provides a description of the wide range of interference occurring in telecommunication systems. The proposed estimation algorithm is based on the multi-Gaussian approximation. The results of the simulation tests showed high efficiency of the method and its low numerical load.*

Streszczenie.** W artykule zaproponowano metodę estymacji sygnału w przypadku gdy wariancja szumu pomiarowego opisana jest rozkładem gruboogonowym. Rozważany model szumu pozwala na opis szerokiego zakresu zakłóceń pojawiających się w systemach telekomunikacyjnych. Proponowana metoda oparta jest na aproksymacji wielogaussowkiej. Wyniki badań symulacyjnych, wykazały wysoką skuteczność proponowanej metody i niskie obciążenie numeryczne. – **Metoda estymacji w obecności szumu o grubo ogonowym rozkładzie wariancji

Keywords: nonstationary noise, heavy-tailed distribution, real time estimation.
Słowa kluczowe: szum niestacjonarny, rozkład gruboogonowy, estymacja w czasie rzeczywistym.

Introduction

Electromagnetic disturbances influencing transmission channel are common issue in industrial electronics and telecommunication systems [1, 2, 3, 4, 5]. Many of the arising problems, as for instance: changes of transmission channel properties, effects of electromagnetic disturbances, fluctuations of reflected radar signal, outlier measurements, intended jamming, may be modelled as modification of measurement noise parameters [6, 7, 8]. These changes has stochastic character and should be described as a process with adequate distribution [9, 10, 11, 12, 13] for instance: Rice, Rayleigh, t-student, K-distribution, Weibull. These are heavy-tailed distributions. As can be seen in Fig. 1, where examples of probability density functions (pdf) of the mentioned distributions are presented, these distributions are characterised by long tails. This feature allows to describe wide range of outlier noise values, which for example are characteristic effects of electromagnetic disturbances. Thus heavy-tails represent the appearance of big values of noise variance, exceeding many times the mean value. This is clearly seen in Fig. 2. Unfortunately in practice, typical linear estimators are usually tuned to mean value of noise parameters. Obviously that leads to exacerbation of estimation performance. Thus a special approach to estimation is needed in such conditions.

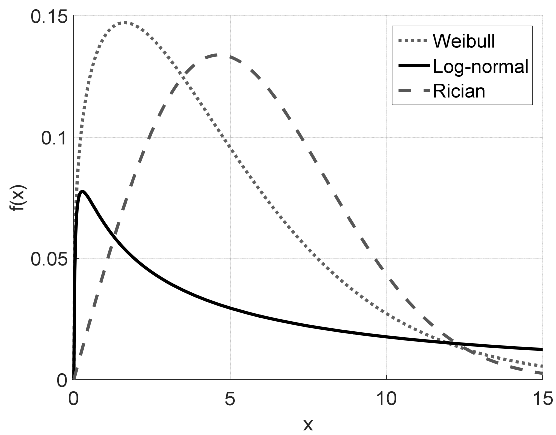

Fig.1. Examples of pdf of heavy-tailed distributions

Estimation method may utilise the multi-Gaussian approximation approach [6]. Its idea is based on the approximation of a given distribution by sum of normal distributions. An example of heavy-tailed distribution approximation is shown in Fig. 3.

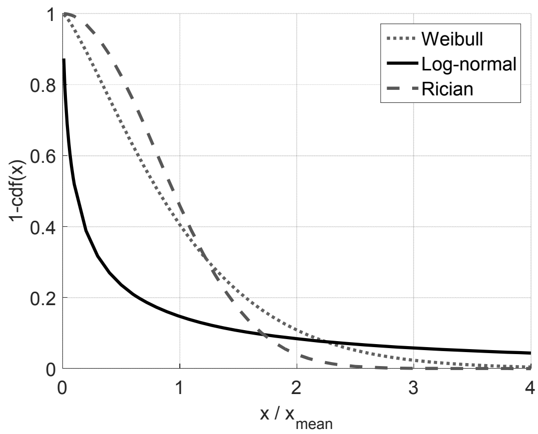

Fig.2. Probability of exceeding the x fold x_{mean} value

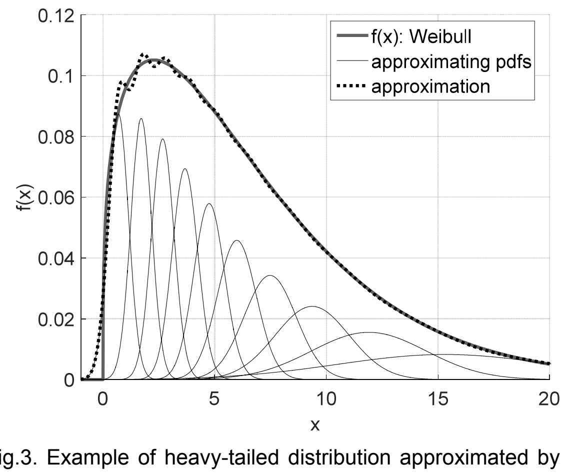

Fig.3. Example of heavy-tailed distribution approximated by set of $M=10$ Gaussian pdfs

Such composition allows to calculate the combined estimate as a weighted sum of estimates obtained in filters constructed for each Gaussian distribution [6, 7, 14, 15], where weights depend on posterior probabilities. This suboptimal estimation approach is flexible and allows to construct reliable estimator for wide variety of noise stochastic characteristics.

The scope of the paper is to propose an estimation algorithm for condition of heavy-tailed distribution of measurement noise variance. The algorithm is designated for real time systems. Estimation algorithm is based on multi-Gaussian approximation of the distribution. The algorithm of adaptive estimator calculates current a posteriori probability of the measurement noise variance level and uses it for adjusting the filter matrix gain. The idea of the proposed method was presented by author in [15] and is described in details in next section. Then, the results of simulation investigations are presented.

Estimation algorithm

The most widely used mathematical model of the diverse systems has a form of state space description, which in case of discrete-time system is the following:

$$x(k+1) = \Phi(k+1,k)x(k) + G_w(k)w(k) \quad (1)$$

where: $x(k)$ is *n* dimensional state vector, $\Phi(k+1,k)$ is the transition matrix, $w(k)$ is white Gaussian sequence with zero mean and covariance matrix $Q(k)$.

The measurement process is modelled by the following observation equation:

$$y(k) = H(k)x(k) + G_v(k)v(k) \quad (2)$$

where: $y(k)$ is *s* dimensional vector containing measurements, $H(k)$ is the observation matrix, $v(k)$ is a zero mean observation noise with a covariance matrix $R(k)$.

For problems stated in the introduction, the estimation algorithm should take into consideration measurements taken in presence of noise with unknown variance sequence $R(k)$. A general approach leads to a solution where the dynamic system state vector estimate $\hat{x}(k/k)$ can be found [6, 14, 16] as a conditional mean of the following form:

$$\hat{x}(k/k) = E[x(k)/Y_1^k, R_1^k] \quad (3)$$

where: $Y_1^k = \{y(1), y(2), ..., y(k)\}$ is the sequence of measurements, $R_1^k = \{R(1), R(2), ..., R(k)\}$ denotes the noise variance sequence.

Unfortunately, as shown in [6, 14] the optimal solution to (3) cannot be implemented in practice, thus some suboptimal solution should be utilized. The proposed solution is based on the multi-Gaussian approximation of heavy-tailed $R(k)$ distribution. In this case the state vector estimate (3) may be expressed as the weighted sum of M partial estimates corresponding to each Gaussian component. Each of these partial estimates can be calculated by prediction of the estimates obtained at the previous time step. Weights depend on the a posteriori probability of validity of certain Gaussian component. The idea of the proposed algorithm has the following form:

$$\hat{x}(k/k) = \sum_{m=1}^{M} \hat{x}_m(k/k, R(k) = R_m(k)) \cdot p(R(k) = R_m(k)/Y_1^k) \quad (4)$$

where: $\hat{x}_m(k/k)$ is partial estimate corresponding to the *m-th* Gaussian distribution. The a posteriori probability $p(R(k) = R_m(k)/Y_1^k)$ will be denoted as $p_m(k)$.

Utilizing the above idea to the Kalman filter method, the suboptimal estimate $\hat{x}(k/k)$ can be formulated as following:

$$\hat{x}(k/k) = \hat{x}(k/k-1) + K_\Sigma(k)[y(k) - H(k)\hat{x}(k/k-1)] \quad (5)$$

where: $\hat{x}(k/k-1)$ is the estimate extrapolation calculated in typical way:

$$\hat{x}(k/k-1) = \Phi(k,k-1)\hat{x}(k-1/k-1) \quad (6)$$

and $K_\Sigma(k)$ is combined gain matrix calculated as weighted sum of partial gains $K_m(k)$:

$$K_\Sigma(k) = \sum_{m=1}^{M} p_m(k) K_m(k) \quad (7)$$

where: $p_m(k)$ is a posteriori probability of validity of the *m-th* approximating Gaussian distribution, $K_m(k)$ is partial gain matrix.

The partial gain matrix $K_m(k)$, which is related to the *m-th* approximating Gaussian distribution, is calculated as follows:

$$\begin{aligned} K_m(k) &= P(k/k-1)H^T(k) \times \\ &\times [H(k)P(k/k-1)H^T(k) + G_v(k)R_m(k)G_v^T(k)]^{-1} \end{aligned} \quad (8)$$

where: extrapolation covariance matrix $P(k/k-1)$ is calculated in typical way:

$$\begin{aligned} P(k/k-1) &= \Phi(k,k-1)P(k-1/k-1)\Phi^T(k,k-1) + \\ &+ G_w(k)Q(k-1)G_w^T(k) \end{aligned} \quad (9)$$

The estimation covariance matrix $P(k/k)$ should take into account information on errors from all M partial estimates, which results in the formula:

$$\begin{aligned} P(k/k) &= P(k/k-1) - K_\Sigma H(k)P(k/k-1) + \\ &+ \sum_{m=1}^{M} p_m(k)[K_m(k) - K_\Sigma(k)] \cdot S(k) \cdot [K_m(k) - K_\Sigma(k)]^T \end{aligned} \quad (10)$$

where: where $S(k)$ is covariance matrix of innovation process: $S(k) = z(k/k-1) \cdot z^T(k/k-1)$

A posterior probabilities $p_m(k)$ depend on the noise stochastic characteristics. When there is no time correlation between consecutive $R(k)$ values, $p_m(k)$ can be found assuming constant a priori probabilities $q_m(k)$ which is shown by (11). In correlated case the Markov chain description should be used [6].

$$p_m(k) = \frac{f(y(k)/R(k) = R_m,\ Y_1^{k-1})\, q_m(k)}{\sum_{i=1}^{M} f(y(k)/R(k) = R_i,\ Y_1^{k-1})\, q_i(k)}; \quad (11)$$

where: $m=1, ..., M$ and

$f(y(k)/R(k) = R_i, Y_1^{k-1}) =$
$N\left[H(k)\hat{x}(k/k-1),\ \ H(k)P(k/k-1)H^T(k) + G_v(k)R_i G_v^T(k)\right]$

denotes the Gaussian probability density function of the predicted estimate.

Simulation results

The performance of the proposed algorithm was investigated in simulations with use of Monte Carlo method with N=10000 runs performed in Matlab environment. The results presented below were carried out for the third-order system describing movement of the object. In this case the state vector $x(k)$ consists of three components: position, velocity and acceleration. The parameters of the equations (1) and (2) were assumed as follows:

$$\Phi = \begin{bmatrix} 1 & T_s & \frac{T_s^2}{2} \\ 0 & 1 & T_s \\ 0 & 0 & 1 \end{bmatrix}, \; G_W = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \; H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \; G_v = 1,$$

and sampling time $T_s=0.1s$. $w(k)$ and $v(k)$ were zero mean Gaussian noises with variances $Q(k)=(0.1)^2$ and $R(k)$ modelled as random variable with Weibull ($A=7, B=1.3$) or log-normal ($\mu=3, \sigma=2$) distribution. $M=10$ Gaussian distributions were used to approximate distribution of $R(k)$ variance.

Fig. 4, 5 and 6 present root mean square error (RMSE) of estimation of three components of state vector $x(k)$ in case of log-normal distribution of $R(k)$ and Fig. 7 in case of Weibull distribution. The performance of the proposed algorithm based on multi-Gaussian approximation (MGAF) was also compared to the linear Kalman filter with constant average $R(k)$ value (KFAR) and with optimal Kalman filter (OKF). Estimation in the latter filter is carried out with full knowledge of the $R(k)$ instantaneous values, which is of course possible in simulations only, but it enables to obtain the lower bound of the estimation accuracy. Next estimation gain $G(k)$ of the proposed algorithm in comparison to KFAR was evaluated. The gain $G(k)$ was defined as following:

$$G(k) = \frac{RMSE_{KFAR}(k) - RMSE_{MGAF}(k)}{RMSE_{KFAR}(k)}$$

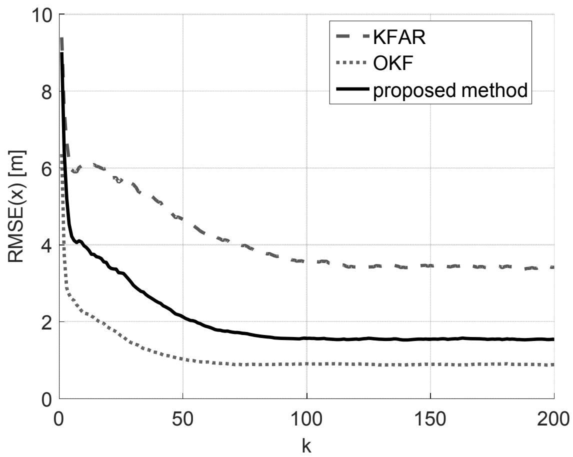

Fig.4. RMSE of position estimation in a case of log-normal distribution of $R(k)$

Fig. 8, 9, 10 show gain $G(k)$ in a case of log-normal and Weibull distribution of $R(k)$ for various number of approximating normal distributions M and for some values of $Q(k)$ to check properties of filters for different relation of $R(k)/Q(k)$.

As can be seen in the Fig. 4, 5, 6, 7 the proposed algorithm reveals good performance. The estimated value has smaller RMS error than linear Kalman filter with constant average $R(k)$ value. The proposed filter does not considerably differ from the optimal filter with known values of the noise variances.

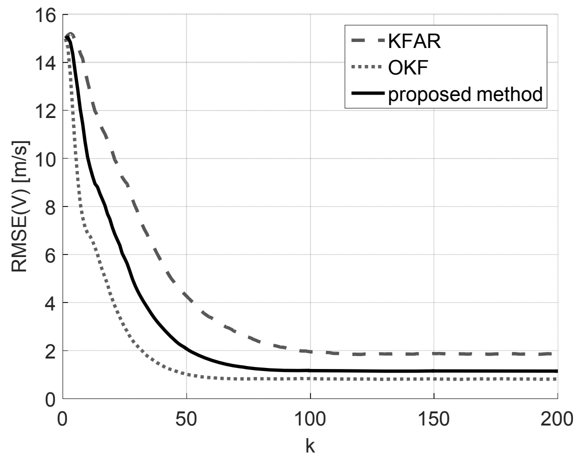

Fig.5. RMSE of velocity estimation in a case of log-normal distribution of $R(k)$

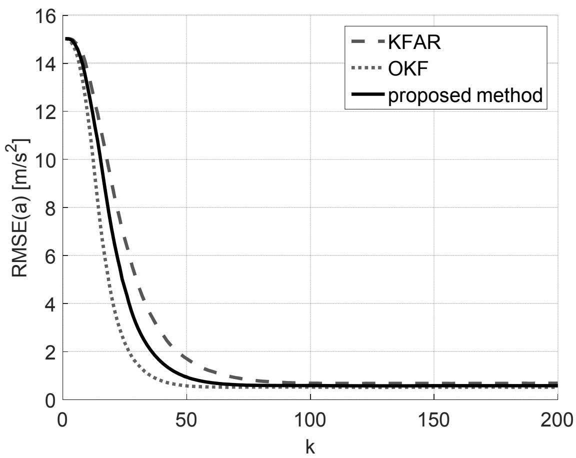

Fig.6. RMSE of acceleration estimation in a case of log-normal distribution of $R(k)$

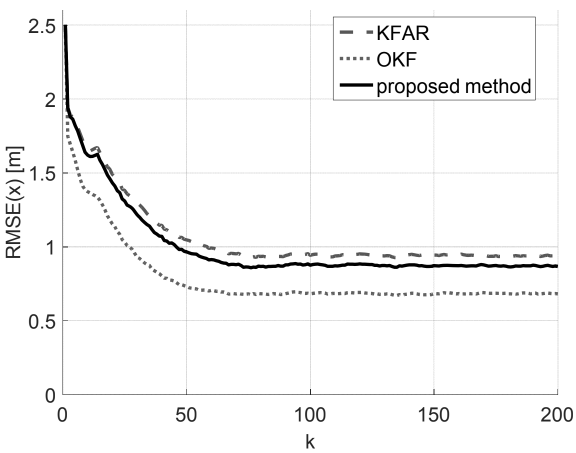

Fig.7. RMSE of position estimation in a case of Weibull distribution of $R(k)$

As it follows from the Fig. 8, 9, 10 the proposed filter reveals better performance than traditional KFAR filter. The performance gain of the method highly depends on the $R(k)$ distribution shape. It increases with increase of relevancy of the distribution tail. Relation of $R(k)/Q(k)$ is also meaningful. The gain may reach up to 60%.

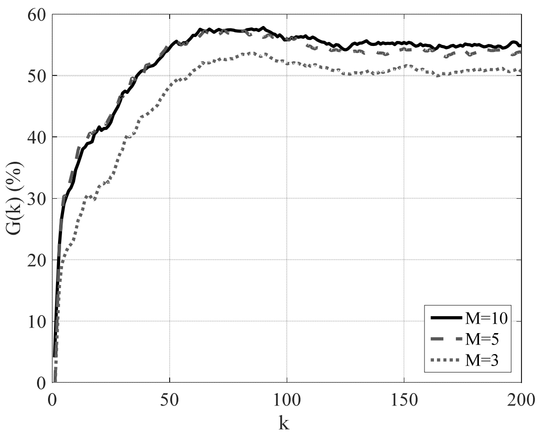

Fig.8. G(k) of estimation in a case of log-normal distribution of $R(k)$ for various M

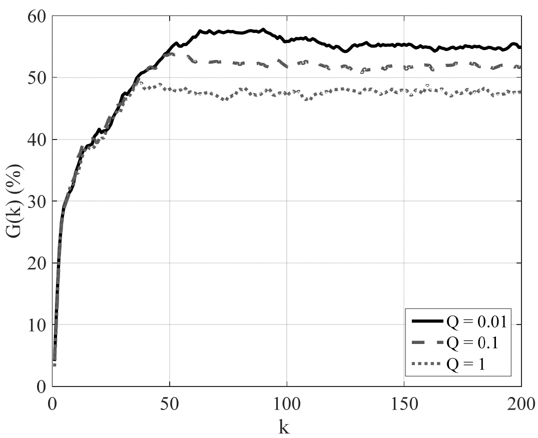

Fig.9. $G(k)$ of estimation in a case of log-normal distribution of $R(k)$ for $M=10$ and various $Q(k)$

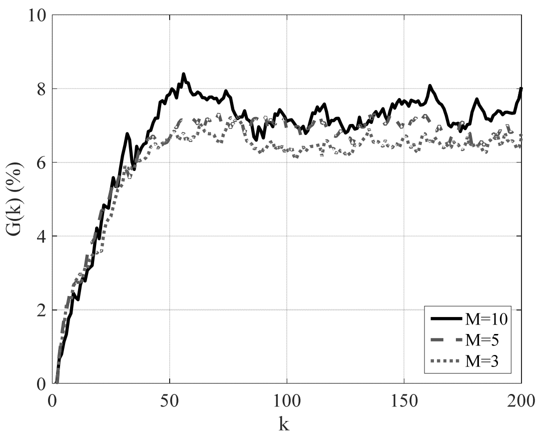

Fig.10. $G(k)$ of estimation in a case of Weibull distribution of $R(k)$ for various M

The number of approximating pdfs also influences performance characteristics. Simulation experiments revealed that the optimal value of M should be selected according to the distribution type and parameters. M should be chosen within the range between 3 and 10.

Conclusions

The paper presents an adaptive estimation method for systems where measurement noise variance is described by heavy-tailed distribution. The proposed algorithm is derived with use of multi-Gaussian approximation of the noise variance distribution. The filter calculates a posteriori probability of the measurement noise variance level and uses it for adjusting the filter matrix gain. The simulation results have shown that the proposed method reveals better performance than Kalman filter that applies average value of observation noise variance. The performance gain increases with increase of relevancy of the distribution tail. The proposed filter does not considerably differ from the optimal filter.

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