Autonomous implicit models of pinched hysterereses with application to memristors

Abstract. This paper analyzes systems with an autonomous dynamics equivalent to that of mem-inductors and mem-capacitors. The systems have oscillatory inductive and capacitive pinched self-crossing hysterereses that follow the folded saddle dynamics around the origin and two equilibria of the center type, each located on the opposite sides of singularity (impasse curve). When parameters of the systems change to yield an increased frequency of oscillations, then the areas enclosed by the hysterereses decrease. The time zero-crossing property of the systems is also discussed.

Streszczenie. W artykule dokonano analizy dynamiki pewnej klasy układów autonomcznych. Rozważane układy charakteryzują się indukcyjnymi i pojemnościowymi, ściśniętymi i przecinającymi się hysterezami, powstającymi w dynamice siodeł na falde z dwoma punktami równowagi typu centrum, położonymi po przeciwnych stronach krzywej osobliwej z punktami impasowymi. Wraz ze zmianą parametrów prowadzące do zwiększenia częstotliwości oscylacji obszar objęty histerezą zmniejsza się. Właściwość jednocześnie zerowania się sygnałów w czasie jest również rozważana.

Keywords: memristive systems, hysteric models, folded saddles, lemniscates

Introduction

Each of the six basic mem-elements is typically described by the input-state-output model \( y = g(x)u, x' = u \), where \( y, x \) and \( u \) stand for the output, state and input variables, respectively [1]-[3]. The output-input characteristics \( y-u \) are of pinched self-crossing type with the \( y \) and \( u \) zero crossing occurring at the same time instants. Moreover, when the frequency of the input \( u \) increases, then the area enclosed by the pinched hystereresis decreases. Also, typically, for the frequency \( f \to \infty \), the double-valued pinched hystereresis property diminishes and the \( y-u \) characteristic tends to that of a single-valued one. Certain physical elements or phenomena may also have hysterereses with decreasing area when \( f \) increases. The \( y-u \) characteristics can also be of a tangential rather than self-crossing nature [4]. The input-state-output models are non-autonomous with \( x \) being a periodic forcing signal (for instance a sinusoidal one). It turns out, however, that the memristive nature and characteristics can also be found in oscillatory circuits with autonomous models. This paper shows two dual circuits as such examples. The most important feature of the circuits is the presence of a folded saddle in the circuits’ two-dimensional phase portraits. However, such a saddle is not an equilibrium point, as it is a phase plane point placed on a singularity manifold. Trajectories can cross the singularity thanks to that folded saddle, a well-known property of implicit systems [5].

Circuits with folded saddles

Three parallel elements, inductor \( L \), capacitor \( C \) and a nonlinear resistor with the voltage-current characteristic \( v = \gamma + x^2, \gamma \in \mathbb{R} \), can be analyzed through the equation

\[
2LC(x_1')^2 + 2LCx_1x_2' + ax_1' = -\gamma - x_2^2
\]

where \( x_1 \) is the current through the nonlinear resistor \((x_1 = i) \) and \( a = L \). A dual version of such a circuit with a series connection of the elements (nonlinear resister has the current-voltage characteristic \( i = \gamma + v^2 \)) is also described by (1) with \( x_1 \) being the voltage across resistor \((x_1 = v) \) and \( a = C \). Equivalently, we have implicit ODEs \( A(x)x' = B(x) \), \( x \equiv (x_1, x_2) \) that describe both circuits, as follows

\[
x_1x_1' = \frac{1}{2 LC} x_2, \quad \frac{a}{a C} x_1x_2' = -\gamma x_1 - x_2^2 - \frac{1}{2b} x_2
\]

where \( x_2 \) is the current through capacitor and \( b = C \) in the parallel circuit, while \( x_2 \) is the voltage across inductor and \( b = L \) in the series circuit. To simplify discussion, let’s assume that \( L = 1, C = 1 \). Then the nature of the point \( (x_1, x_2) = (0,0) \), which is located on the fold line \( x_1 = 0 \), depends on the value of \( \gamma \). Notice that \( A(0) \) is singular, and \( b(0) = 0 \). The origin is a folded saddle for \( \gamma > 0 \), folded saddle-node for \( \gamma = 0 \), folded node for \( 0 < \gamma < 1/8 \), degenerate folded node for \( \gamma = 1/8 \), and a folded focus for \( \gamma > 1/8 \). There are trajectories crossing the fold line \( x_1 = 0 \) in the first four cases. No trajectory crossing \((0,0)\) exists in the folded focus case.

Fig.2. Two dual mem-circuits described by the autonomous implicit model: \( x_1x_1' = (1/2b)x_2, ax_2' = -\gamma - x_2^2 \) (modified (2)).

Fig.1(a) shows a phase portrait for \( \gamma = -1 \) with a folded saddle at the origin. Trajectory crosses the origin at \( dx_1/dx_2 = \{-1/2, 1\} \). A stable and unstable equilibria exist at \((\sqrt{\gamma}, 0)\) and \((-\sqrt{\gamma}, 0)\), respectively, both with complex eigenvalues. Notice also that no periodic solution exists for any value of \( \gamma \) in the two circuits. The lack of periodic solution is changed if a modification of the circuit is introduced by adding the elements controlled by \( x_1' \) as shown in Figs.2(a),2(b). Such a modification yields a rather dramatic change in the dynamics of the circuits and, as shown in this paper, makes that the added elements have the characteristics of mem-elements. In particular, the added element in Fig.2(a) has a pinched characteristic \( \psi - x_3 \) of a meminductor.

Fig.1. Solutions of implicit autonomous models with parameters \( a = b = 1, \gamma = -1 \). Solutions cross the origin at the directions indicated by the dashed lines. Except for the origin, the fold curve \( x_1 = 0 \) consists of impasse points where trajectories terminate at or originate from. Two equilibria exist also at \((\pm 1, 0)\).
Fig. 3. Periodic solutions crossing the origin of the modified circuit in Fig.2(a): (a) $L = 1/4$ (dashed), $L = 1/2$, $L = 3/4$, $L = 1$, $L = 5/4$ (smallest loop), (b) $L = 4$ (dashed), $C = 2$, $C = 1/2$, $C = 1/8$, $C = 1/32$ (smallest loop).

(ψ=flux, $x_{3}$=current), while the added element in Fig.2(b) has a pinched characteristic $q-x_{3}$ of a memcapacitor ($q$=charge, $x_{3}$=voltage). The above modifications yield the model (1) without the $ax_{1}^{2}$ term. This results in the second equation in (2) being modified to

$$ax_{1}^{2} = -\gamma - x_{1}^{2}$$

while the first equation in (2) does not change. There is also a change in the nature of the two equilibrium points ($\pm \sqrt{-\gamma}$, 0) in Fig.1(a), which now becomes centers with purely imaginary eigenvalues, as shown in Fig.1(b). The nature of the folded point $(0, 0)$ has not changed. The origin still remains a folding saddle with two trajectories crossing the fold $x_{1} = 0$ from one side to the other. The above facts result in the pinched trajectories moving between the sides of $x_{1} = 0$ and encircling the centers at $(\pm \sqrt{-\gamma}, 0)$. Thus the $x_{1}$-$x_{2}$ trajectories are symmetrical as shown in Figs.3(a),3(b) for several values of $L$ (= $a$) and $C$ (= $b$). It can be shown for both circuits that the trajectories cross the origin at angles $\pm \frac{\sqrt{2\gamma}}{a}$ (\(= \kappa\)). Also, the period of oscillation of $x_{1}$ is $T = 2\pi \sqrt{ab}$ and, thus, the period of $x_{2}$ is $\pi \sqrt{ab}$, half the period of $x_{1}$.

**Theorem 1:** The pinched trajectory $(x_{1}, x_{2})$ is such that the variables $x_{1}$ and $x_{2} \pm 2\sqrt{2\gamma/a}$, $\gamma < 0$, satisfy the following equation

$$x_{1}^{4} = c^{2}(x_{1}^{2} - x_{2}^{2})$$

with $c^{2} = -2\gamma > 0$.

**Proof:** The first equation in (2) and (3) yield $ax_{2}dx_{2} = -2(\gamma + x_{1}^2)bx_{2}dx_{1}$, which, after integration takes the form $ax_{2}^{2} = -2bx_{1}^{2} - bx_{1}^{2}$. From the fact that the trajectory is pinched at $(0, 0)$ we obtain the constant of integration to be zero. Now, with the scaling $x_{2} \rightarrow x_{2} \sqrt{-2\gamma/a}$ (or equivalently $x_{2}^{2} \rightarrow x_{2}^{2}(-2\gamma/a)$) we obtain $ax_{2}^{2}(-2\gamma/a) = -2bx_{1}^{2} - bx_{1}^{2}$, which is equivalent to (4).

Since (4) is the lemniscate of Gerono [6] (see Appendix I), therefore the pinched hysteresis $(x_{1}, x_{2})$ of the two circuits is a scaled version of that lemniscate. In particular, the properties of the lemniscate of Gerono are modified by the factor of $\kappa$, which appears in the above theorem. This scaling factor applies to such properties of the lemniscate of Gerono as the area enclosed by the lemniscate, arc length, parameterized equations, polar formula and others. As shown above, variables $x_{1}$ and $x_{2}$ satisfy the following equation $ax_{2}^{2} = -2bx_{1}^{2} - bx_{1}^{2}$. One possible parameterization of such an equation is

$$x_{1}(t) = \sqrt{-2\gamma \sin \left(\frac{t}{\sqrt{ab}}\right)}, \quad x_{2}(t) = -\gamma \sqrt{\frac{b}{a}} \sin \left(\frac{2t}{\sqrt{ab}}\right)$$

for $0 \leq t < 2\pi \sqrt{ab}$ (one complete loop of hysteresis).

Fig. 4. Hystereses -- $Lx_{1}$ versus $x_{3}$ for the circuit in Fig.2(a): (a) $L = 1/4$ (dashed), $L = 1/2$, $L = 3/4$, $L = 1$, $L = 5/4$ (largest loop), (b) $L = 4$ (dashed), $C = 2$, $C = 1/2$, $C = 1/8$, $C = 1/32$ (smallest loop).

Using the above parametric equations one can show that the area enclosed by a complete loop (two lobes) is

$$A = 2\int_{0}^{\pi/\omega} \sqrt{\frac{\pi}{\omega}} \sin \left(\frac{\theta}{\sqrt{ab}}\right) \cos \left(\frac{\theta}{\sqrt{ab}}\right) d\theta = -\kappa \frac{\pi}{2\omega}.$$  

Notice that the value of $A$ in (6) is a scaled version of the corresponding value of the area of lemniscate of Gerono, $A_{G} = \frac{\pi^{2}}{8}$ [6]. Thus we have $A_{G} = -\frac{\pi}{2\omega}$ (since $c^{2} = -2\gamma$ in (6)). This gives $A = \kappa A_{G}$.

One method of computing the arc length of the pinched hysteresis is to use parametric equations (5) to have

$$s = 4 \int_{0}^{\pi/(2\omega)} \sqrt{(x_{1}'^{2}) + (x_{2}')^{2}} dt = 4 \int_{0}^{\pi/\omega} \sqrt{\frac{2b}{a} \left(\frac{2b}{a} \frac{b}{a} - \frac{b}{a} \frac{1}{a} \right) \sin^{2} (\omega \theta) + \frac{16\gamma^{2}}{a^{2}} \sin^{4} (\omega \theta) dt}$$

with $\omega = 1/\sqrt{ab}$. If we use the modified coordinates $(x_{1}, x_{2} \sqrt{-2\gamma/a})$ with $x_{1}$ and $x_{2}$ as in (5), then the above formulas simplify as $ax_{2}^{2} = -2bx_{1}^{2} - bx_{1}^{2}$ becomes the lemniscate of Gerono. Notice the dependence of the areas of hystereses shown in Figs.3(a),3(b) on $L$ and $C$, which determine frequency of oscillations $f$. This is discussed in more detail in the next section where an interesting interpretation of the $-ax_{1}$-$x_{2}$ trajectory of the controlled elements is provided. With $x_{3} = -ax_{1} - x_{2}$, the $-ax_{1}$-$x_{2}$ characteristic has a typical interpretation of a mem-inductor and a mem-capacitor for the circuits in Figs.2(a) and 2(b), respectively.

**Controlled elements as mem-inductors and mem-capacitors and their properties**

The current through the controlled element and through $L$ in Fig.2(a) is $x_{3}(t)$ and the controlled element’s voltage is $V(t) = -Lx_{1}'(t)$. Thus, the integral of $V(t)$ is the flux of $V(t)$ is $\int V(t) dt = -Lx_{1}(t) + const$. Assuming, $const = 0$, the $-Lx_{1}(t)$ versus $x_{3}(t)$ graph is actually the flux-current characteristic of the controlled element. Similarly, in Fig.2(b) we have that the voltage across the controlled element and across $C$ is $x_{3}(t)$ and the controlled element’s current is $I(t) = -Cx_{1}'(t)$. Thus, we have $q(t) = \int I(t) dt = -Cx_{1}(t) + const$. Again, for $const = 0$, the $-Cx_{1}(t)$ versus $x_{3}(t)$ graph is actually the charge-voltage characteristic. Figs.4(a),4(b) show typical graphs of $-Lx_{1}$ versus $x_{3}$ obtained (for circuit in Fig.2(a)) by solving the first equation in (2) with (3) for several values of $L$, $C$ and two values of $\gamma$. Figs.5(a),5(b) show the time responses of $x_{3}(t)$ and $-Lx_{1}(t)$ for two sets of parameters. Note that the areas of hystereses in Fig.4(a),4(b) can be computed with
with $c$.

The oscillatory trajectory of (9), $x_3 = -x_1 - x_2$ as follows

$$A = 2a \int_0^{\pi \sqrt{ab}} (-x_1 - x_2) (-x_1') dt$$

$$= 2a \int_0^{\pi \sqrt{ab}} (x_1 x_1' + x_2 x_2') dt$$

$$= 2a (x_1^2/2) \int_0^{\pi \sqrt{ab}} x_2 x_2' dt$$

$$= -\frac{16a\gamma}{3} \sqrt{-2x_1} x_2 = -\frac{8a\gamma}{3}$$

where we used the fact that $x_1^2/2 \int_0^{\pi \sqrt{ab}} x_2 x_2' dt = 0$ for $x_1(t)$ in (5) and $\int_0^{\pi \sqrt{ab}} x_2 x_2' dt$ is computed in (6). Clearly, with increasing frequency $f$ the areas of hysteresis in Fig.4(a),4(b) decrease. This is a well-known fingerprint of most memristive elements. Also, the identical zero time crossing property of hysteresis $-ax_1 x_3$ is as follows. Transforming graphs from the $(x_1, x_2)$ coordinate system into $(x_1, -ax_1)$ changes the line $x_1 = -x_2$ into $x_1 = 0$ (since $x_3 = -x_1 - x_2$), while the line $x_1 = x_2$ changes into $-ax_1 = ax_2/2$. Thus, the dotted lines in Figs.3(a),3(b), being the tangential lines for the loops with $L = \frac{1}{2}, C = \frac{1}{2}$ (Fig.3(a)) and $L = 1, C = 2$ (in Fig.3(b)), are changed to the dotted lines $-Lx_1 = \frac{1}{2}x_3$ in Fig.4(a)) and $-Lx_1 = \frac{1}{2}x_3$ (in Fig.4(b)). The pinched hysteresis loops inside the sector determined by the dotted lines have the property that the zero crossings of $-Lx_1$ and $x_3$ occur at the same time instants. Hysteresis loops not entirely inside that sector, that is the dashed curves in Figs.4(a),4(b), do not have the zero crossing property. Those curves correspond to the dashed curves in Figs.3(a),3(b) intersecting the straight dotted lines $x_1 = \pm x_2$. The condition for the trajectories $-ax_1, x_3$ to have the zero crossing property for both mem-circuits is $\kappa \leq 1$.

**Pinched hysteresis due to Devil's curve**

In this subsection, we discuss another modification of (2) yielding a hysteresis equivalent to Devil's curve [7]. Consider the following implicit equations

$$x_1 x_1' = \frac{1}{2ab} (ax_2 - 2bx_2^3)$$

$$x_2 x_2' = -\gamma - x_1^2$$

**Theorem 2:** The oscillatory trajectory of (9), $x_1$ versus $x_2$, is pinched at the origin for $\gamma < 0$ and $ab/2 > 0$, and satisfies the following equation

$$x_1^4 - x_2^4 = c^2 x_1^2 - d^2 x_2^2$$

with $c^2 = -2\gamma > 0$ and $d^2 = ab/2 > 0$.

**Proof:** From (9) we obtain $(ax_2 - 2bx_2^3)dx_2 = -2(\gamma + x_1 x_1')dx_1$, which, after integration takes the form $ax_2^2 - bx_2^4 = -2b\gamma x_1 x_1' - bx_1^4$. From the fact that the trajectory is pinched at (0) we obtain the constant of integration to be zero. Simple rearrangement of terms and division by $b$ in the last equation yield $x_1^4 = -x_2^4 = -2(\gamma + ab)x_2^2/2$. This is equivalent to (10).

![Devil's curves](image)

**Fig. 5.** Periodic solutions of $-Lx_1$ and $x_3$ for the dotted hystereses: (a) in Fig.4(a), (b) in Fig.4(b).

![Devil's curves](image)

**Fig. 6.** (a) Devil's curves for various values of $c^2 = ab/2$ and $d^2 = 1$ (see (10)). (b) Pinched hysteresis passes through the origin as Devil's curve (10) with $c^2 = 1$ and $d^2 = 4$. (c) Phase portrait with 4 saddles (marked by the characters) away from the fold $x_1 = 0$, folded saddle and two folded centers at $x_1 = 0$. Two trajectories cross the fold line along the lines $x_1 = \pm x_2$. This is the degenerate Devil's lemniscate (10) with $c^2 = d^2$, yielding $x_1^2 + x_2^2 = c^2$ in general, and with $c^2 = 1$ in the case illustrated in this plot. (d) Phase portrait of the modified system (with (11)) resulting in a pinched hysteresis through the origin. No other trajectories crossing the fold $x_1 = 0$ exist. The $\gamma = -1/2$ in all the cases above.

The graph of (10) is known as Devil's curve (or lemniscate) [7] and is shown for several parameters of $a, b$ and $\gamma$ in Fig.6(a) (in all cases the condition $c/d = \sqrt{-2\gamma/b} > 1$ is satisfied). When the ratio c/d is less than 1, then the graphs in Fig.6(a) are rotated by $90$ degrees. Each Devil's curve is of degree 4, has a cuspode at the origin and the genus is equal 2. This does not allow for a birational parametrization. Other properties of the Devil's curve (area enclosed, parametrization, arc length, Abelian integrals, etc.) can be found in [7, 8]. Fig.6(b) shows a phase portrait with a hysteresic solution in the form of Devil's curve tangential to the lines $x_1 = \pm x_2$ at the origin. In the case $-2\gamma = \gamma/2$ we obtain a degenerate Devil's curve - a circle. Such a case is shown in Fig.6(c). Two solutions along the straight lines $x_1 = \pm x_2$ pass through the folded saddle at the origin.

A modification, for example when the parameters $a, b$ or $\gamma$ bifurcate, is possible. This will yield a distorted Devil's curve equation, however, still yielding a pinched hysteresis at the origin. For example, by changing the first equation in (9) to

$x_1 x_1' = \frac{1}{2ab} (ax_2 - 2bx_2^3)$

and keeping the second equation in (9) unchanged, we obtain the phase portrait as in Fig.6(d), where the four saddles (marked by $\circ$) do not result in a folded saddle at the origin anymore (as they do in Fig.6(c)). The folded saddle still remains intact (as in Fig.6(b)), allowing for a pinched hysteresic solution to travel through the origin in a periodic manner. No other periodic trajectory moving between different sides of the fold $x_1 = 0$ exists. It is a straightforward exercise to show that the hysteresis is described by (10) with the term $x_1^2$ replaced by $x_1^2/2$. 

![Devil's curves](image)
Further analysis of the hysteresis in the form of Devil’s curves and their transformations into the pinched, same time instant zero crossing memristive-like hystereses in the first and third quadrants are possible by defining a new variable, say $x_3 = x_1 - x_2$ in a way similar to that discussed earlier for the lemniscate of Geron. Finally, it is worth pointing out that the pinched oscillatory hystereses considered in this paper are closely linked to planar conservative dynamical systems which are well researched and found many applications [9]. The following lemma links Devil’s curve with the well-known Duffing’s autonomous and undamped equation.

**Lemma:** The variable $x_2$ in (10), being a solution of (9) with the initial conditions $x_1(0)$ and $x_2(0)$, satisfies the following Duffing’s equation $ax_2'' = -\frac{1}{b}x_2 + \frac{1}{a}x_3^3$, with the conditions $x_2(0)$ and $x_2'(0) = -\left(\gamma + \frac{1}{2}x_1^2\right)/a$.

**Proof:** Duffing’s equation for $x_2(t)$ can be easily obtained by differentiating the second equation in (9) and using the first equation to replace the term $x_1 x_2'$ obtained after differentiation. The initial condition $x_2'(0)$ follows from the second equation in (9) at $t = 0$.

Parameters $b$ and $a$ control the linear stiffness and amount of non-linearity in the restoring force, respectively. The right-hand side in the above Duffing’s equation, $(-1/b)x_2 + (2/a)x_3$, is the restoring force $F(x_2)$, provided by the non-linear spring, as we have $m x_2'' = F(x_2)$ with $m = a$.

Obtaining pinched hystereses numerically: a word of caution

All hystereses and time-series plots in this paper were created using Maple 18. Special care should be exercised in order to obtain hystereses. Some extra analysis of the directions at which a hysteresis crosses the origin is needed. Also, choosing the right numerical solver is important, as is the selection of that solver’s optional parameters. For example, the Gear extrapolation method in Maple 18 that was used in solving the autonomous ODEs in this paper, has two choices of the extrapolation procedure (Burlirsch-Stoer or polynomial extrapolation), each with about 18 optional parameters [10]. The Livermore stiff ODE solver lsode has 8 choices for submethods, each with 16 optional parameters. The partial code in Appendix II illustrates how to successfully obtain both the pinched hysteresis in Fig.6(b) and the corresponding time-series solutions $x_1(t)$ and $x_2(t)$ by using the Gear method (with the default Burlirsch-Stoer rational extrapolation procedure) and only two optional parameters, **abserr** and **reerr**, specifying the desired accuracy of the solution.

**Conclusion**

The dynamics of the autonomous, differentiable and implicit models analyzed in this paper follow either the scaled lemniscate of Geron (with the $k = \sqrt{-2} \gamma b/a$ parameter) or Devil’s curve. Both planar curves yield pinched, self-crossing hystereses resulting from a folded saddle located at a fold (singularity). The hystereses’ areas of the controlled elements in the two dual autonomous circuits (associated with Geron’s lemniscate) decrease with an increased frequency.

**Appendix II: Maple code**

```maple
with(plots): g:=0.5: a:=4: b:=1:
eq1:=diff(x1(t),t)=(a-x2(t)-2-b*x2(t)^3)/(2-a*b*x1(t)): eq2:=a*diff(x2(t),t)=g-x1(t)^2:
out:=dsolve([eq1,eq2,x1(0)=0.2e-5,x2(0)=0.1e-5],[x1(t),x2(t)],type=numeric,
             method=gear,abserr=10^(-10),reerr=10^(-10)):
do(solve,[[t,x1(t)],[t,x2(t)]],-20..20, numpoints=5000)
do(solve,[[x2(t),x1(t)],[t,x1(t)]],-20..20, numpoints=5000,color=black,axes=boxed,
thickness=2,view=[-0.55..0.55,-1.25..1.25],font=[label,"HELVETICA",20])
```

Analysis of the same time zero crossing property for the two circuits is also presented. The approach presented in this paper is based on autonomous models and it differs from the commonly used input-state-output non-autonomous model approach. Further extensions of the results in this paper to include autonomous dynamical models related to the lemniscates of Bernoulli and Booth seem also be possible.

**Authors:** Wieslaw Marszalek, Rutgers University, Hill Center for Mathematical Sciences, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA, email: w.marszalek@rutgers.edu. Research done at the Opole University of Technology in Poland. Financial support by the Fulbright Foundation in Washington, DC, USA is greatly appreciated.

**REFERENCES**


**Appendix I: Lemniscate of Geron**

Properties of the lemniscate of Geron can be examined from the point of view of analytical geometry in Maple by using its algcurves package [10]. The basic properties are [11]:

- **Parameterization:** $x_1 = \frac{\cos^2(t)}{\sin(t)}$, $x_2 = \frac{\sin^2(t)}{(\cos(t)+1)}$,
  $-\infty < u < \infty$. This parameterization results from the fact that the lemniscate of Geron is of zero genus.
- **Arc length:**
  \[ s = 4c\int_0^1 \sqrt{\frac{3u^2+5u^2+2}{1-u^2}} \, dt \]
  $\approx 4c\int_0^{\pi/2} \sqrt{4\sin^4(t) - 5\sin^2(t) + 2} \, dt$
  $\approx 2c\int_0^{\pi/2} \sqrt{2 + \cos(2t)} + \cos(4t) \, dt$
- **Polar equation:** $r^2 = c^2 \sec^4(t) \cos(2t)$
- **Curvature:** $k(t) = \frac{c^2 \sec^4(t) \sin(t)}{c^2(\cos(2t) + \cos(4t))^2}$