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Tadeusz KACZOREK

Bialystok University of Technology

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Dedicated to Professor Tadeusz Cholewicki on the occasion of his 30 years death anniversary

Normal positive linear systems and electrical circuits

***Abstract**. The notion of normal positive electrical circuits is introduced and some their specific properties are investigated. New state matrices of positive linear systems and electrical circuits are proposed and their properties are analyzed. It is shown that positive electrical circuits with diagonal state matrices are normal for all values of resistances, inductances and capacitances*

***Streszczenie.** W artykule zaproponowano pojęcie dodatniego obwodu elektrycznego oraz przeanalizowano specjalne własności dodatnich układów i obwodów elektrycznych. Wykazano, że dodatnie obwody elektryczne z diagonalnymi macierzami stanu są zawsze normalne dla wszystkich wartości rezystancji, indukcyjności i pojemności. (**Normalne dodatnie układy liniowe i obwody elektryczne**).*

Keywords: normal, positive, linear, system, electrical circuit.
Słowa kluczowe: układ normalny, dodatni, liniowy, obwód elektryczny.

Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in the monographs [2, 15]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The notions of controllability and observability have been introduced by Kalman in [28, 29] and they are the basic concepts of the modern control theory [1, 7, 8, 11, 12, 20, 27, 31]. The controllability, reachability and observability of linear systems and electrical circuits have been investigated in [9, 10, 16, 18, 19, 30]. The asymptotic stability of positive standard and fractional linear systems has been addressed in [6, 15, 26].

Cholewicki has been the pioneer in Poland of the application of the theory of matrices in the analysis and synthesis of electrical circuits [3, 4, 5].

The specific duality and stability of positive electrical circuits have been analyzed in [21] and positive systems and electrical circuits with inverse state matrices in [17]. The stability of continuous-time and discrete-time linear systems with inverse state matrices has been investigated in [25]. The reduction of linear electrical circuits with complex eigenvalues to linear electrical circuits with real eigenvalues has been considered in [24].

Standard and positive electrical circuits with zero transfer matrices have been investigated in [22] and the normal positive electrical circuits have been introduced in [13].

In this paper the normal positive linear systems and electrical circuits are investigated.

The paper is organized as follows. In section 2 some preliminaries concerning positive linear continuous-time systems are recalled. Some properties of the transfer matrices of positive linear systems are presented in section 3. Normal positive linear systems are analyzed in section 4. Normal positive linear electrical circuits are introduced and investigated in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

Preliminaries

Consider the continuous-time linear system

(1a) $\dot{x}=Ax+Bu$,

(1b) $y=Cx$,

where $x=x(t)\in\Re^n$, $u=u(t)\in\Re^m$, $y=y(t)\in\Re^p$ are the state, input and output vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $C\in\Re^{p\times n}$.

Definition 1. [15] The linear system (1) is called (internally) positive if $x(t)\in\Re_+^n$ and $y(t)\in\Re_+^p$, $t\ge 0$ for any initial conditions $x_0\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\ge 0$.

Theorem 1. [15] The linear system (1) is positive if and only if

(2) $A\in M_n$, $B\in\Re_+^{n\times m}$, $C\in\Re_+^{p\times n}$.

Definition 2. [15] The positive linear system (1) for $u(t)=0$ is called asymptotically stable if

(3) $\lim_{t\to\infty} x(t)=0$ for all $x_0\in\Re_+^n$.

Theorem 2. [15] The positive linear system (1) for $u(t)=0$ is asymptotically stable (the matrix A is Hurwitz) if and only if all coefficients of the characteristic polynomial

(4) $p_n(s)=\det[I_n s-A]=s^n+a_{n-1}s^{n-1}+...+a_1 s+a_0$

are positive, i.e. $a_k > 0$ for $k = 0,1,...,n-1$.

We shall consider the positive system (1) with the matrix A of the form

(5)
$$A_1 = \begin{bmatrix} -s_1 & a_1 & 0 & \cdots & 0 & 0 \\ 0 & -s_2 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -s_{n-1} & a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & -s_n \end{bmatrix} \text{ or}$$
$$A_2 = \begin{bmatrix} -s_1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & -s_2 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & -s_3 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & -s_n \end{bmatrix},$$
$$a_k > 0,\ k = 1,...,n-1.$$

Theorem 3. [15] The positive system with (5) is asymptotically stable if and only if

(6) $\quad \mathrm{Re}\, s_k < 0$ for $k = 1,...,n$.

Definition 3. [15] The positive system (1) is called reachable in time $[0, t_f]$ if for any given final state $x_f \in \Re_+^n$ there exists an input $u(t) \in \Re_+^m$ for $t \in [0, t_f]$ that steers the state $x(t)$ from zero initial state $x(0) = 0$ to the final state x_f, i.e. $x(t_f) = x_f$.

Definition 4. [15] A real matrix $A \in \Re_+^{n\times n}$ is called monomial if each its row (column) contains only one positive entry and the remaining entries are zero.

Theorem 4. [15] The positive system (1) is reachable if the matrix

(7) $$R_f = \int_0^{t_f} e^{A\tau} B B^T e^{A^T \tau} d\tau,\ t_f > 0$$

is monomial.

The input

(8) $$u(t) = B^T e^{A^T (t_f - t)} R_f^{-1} x_f \in \Re_+^{mx1},\ t \in [0, t_f]$$

steers the state $x(t)$ of the system from $x(0) = 0$ to $x(t_f) = x_f$.

The positive system (1) is reachable in time $[0, t_f]$ if and only if $A \in M_n$ is diagonal and $B \in \Re_+^{n\times m}$ has m linearly independent monomial columns.

Definition 5. [15] The positive system (1) is called observable in time $[0, t_f]$ if knowing the output $y(t) \in \Re_+^p$ and the input $u(t) \in \Re_+^m$ it is possible to find the unique initial condition $x(0) \in \Re_+^n$.

Theorem 5. [26] The positive system (1) is observable in time $[0, t_f]$ if the matrix

(9) $$O_f = \int_0^{t_f} e^{A^T \tau} C^T C e^{A\tau} d\tau,\ t_f > 0$$

is monomial.

Transfer matrices of positive linear systems

The transfer matrix of the positive linear system (1) is given by

(10) $$T(s) = C[I_n s - A]^{-1} B \in \Re^{p\times m}(s),$$

where $\Re^{p\times m}(s)$ is the set of $p \times m$ rational matrices in *s*.

Theorem 6. If the matrix $A \in M_n$ given by (5) is asymptotically stable (Hurwitz) and $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$ then all coefficients of the transfer matrices

(11) $$T_1(s) = C[I_n s - A_1]^{-1} B,\ T_2(s) = C[I_n s - A_2]^{-1} B$$

are nonnegative.

Proof. If A_1 is Hurwitz and $a_k > 0$, $k = 1,...,n-1$ then the entries of the inverse matrix (12) are rational functions with nonnegative coefficients.

$$[I_n s - A_1]^{-1} = \begin{bmatrix} s+s_1 & -a_1 & 0 & \cdots & 0 & 0 \\ 0 & s+s_2 & -a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s+s_{n-1} & -a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & s+s_n \end{bmatrix}^{-1}$$

(12) $$= \frac{1}{(s+s_1)(s+s_2)...(s+s_n)} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1,n-1} & \alpha_{1,n} \\ 0 & \alpha_{22} & \alpha_{23} & \cdots & \alpha_{2,n-1} & \alpha_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1,n-1} & \alpha_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & \alpha_{n,n} \end{bmatrix},$$

$$\alpha_{11} = (s+s_2)...(s+s_n),\ \alpha_{12} = a_1(s+s_3)...(s+s_n),\ \alpha_{13} = a_1 a_2 (s+s_4)...(s+s_n),$$
$$\alpha_{1,n-1} = a_1...a_{n-2}(s+s_n),\ \alpha_{1,n} = a_1 a_2 ... a_{n-1},$$
$$\alpha_{22} = (s+s_1)(s+s_3)...(s+s_n),\ \alpha_{23} = a_2(s+s_1)(s+s_4)...(s+s_n),$$
$$\alpha_{2,n-1} = a_2...a_{n-2}(s+s_1)(s+s_n),\ \alpha_{2,n} = a_2...a_{n-1}(s+s_1)(s+s_n),$$
$$\alpha_{n-1,n-1} = (s+s_1)...(s+s_{n-2})(s+s_n),\ \alpha_{n-1,n} = a_{n-1}(s+s_1)...(s+s_{n-2}),$$
$$\alpha_{n,n} = (s+s_1)(s+s_2)...(s+s_{n-1})$$

Therefore, if $B \in \Re_+^{n\times m}$ and $C \in \Re_+^{p\times n}$ then all coefficients of the transfer matrix $T_1(s)$ are nonnegative.

The proof for $T_2(s)$ is similar (dual). □

Example 1. Consider the transfer function of the positive system (10) with

(13) $$A = A_1 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C = [1 \quad 3 \quad 2].$$

In this case using (10) and (13) we obtain

$$T_1(s) = C[I_3 s - A_1]^{-1} B$$
$$= [1 \quad 3 \quad 2]\begin{bmatrix} s+1 & -2 & 0 \\ 0 & s+2 & -1 \\ 0 & 0 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(14) $$= \frac{1}{(s+1)(s+2)(s+3)}[1 \quad 3 \quad 2]$$
$$\times \begin{bmatrix} (s+2)(s+3) & 2(s+3) & 2 \\ 0 & (s+1)(s+3) & s+1 \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$= \frac{2s^2 + 9s + 9}{s^3 + 6s^2 + 11s + 6}.$$

The transfer function is minimal-phase since its zeros $z_1 = -1.5$, $z_2 = -3$ are negative. After cancellation of the zero $z_2 = -3$ with the pole $s_3 = -3$ we obtain

(15) $$T_1(s) = \frac{2s+3}{s^2 + 3s + 2}.$$

It is easy to check that if

(16) $$A_1 = \begin{bmatrix} -1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix},\ B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ C_1 = [1 \quad 2 \quad 1]$$

then

(17) $$T_1(s) = C_1[I_3 s - A_1]^{-1} B_1$$
$$= [1 \quad 2 \quad 1]\begin{bmatrix} s+1 & -2 & 0 \\ 0 & s+2 & -1 \\ 0 & 0 & s+3 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{s+1}.$$

In this case we have

(18) $$\mathrm{rank}[B_1 \quad A_1 B_1 \quad A_1^2 B_1] = \mathrm{rank}\begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & -5 \\ 1 & -3 & 9 \end{bmatrix} = 3 = n$$

and

(19) $$\mathrm{rank}\begin{bmatrix} C_1 \\ C_1 A_1 \\ C_1 A_1^2 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix} = 1 < n = 3.$$

Therefore, the standard pair (A_1, B_1) is controllable, but the pair (A_1, C_1) is unobservable.

Consider the SISO (single-input ($m=1$) single-output ($p=1$)) positive linear system with A_1 given by (5) and

(20) $$B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \Re_+^n,\ C_1 \in \Re_+^{1\times n}.$$

It is easy to check that

(21) $$\mathrm{rank}[B_1 \quad A_1 B_1 \quad \cdots \quad A_1^{n-1} B_1] = n \text{ if } a_k > 0,\ k = 1,\dots,n-1.$$

Let z_1, z_2, …, z_{n-1} be the zeros (the roots of $n(s) = 0$) and p_1, p_2, …, p_n the poles (the roots of $d(s) = 0$) of the transfer function

(22) $$T_1(s) = C_1[I_n s - A_1]^{-1} B_1 = \frac{n(s)}{d(s)}.$$

Theorem 7. If

(23) $$\mathrm{rank}\begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{n-1} \end{bmatrix} < n$$

then at least one zero of (22) is equal to its poles.

Proof. It is well-known that if (23) holds then the zeros and poles cancellation occurs in (22). It happens only if at least one zero of (22) is equal to its poles. □

Now let us consider the SISO positive system with A_2 given by (5) and

(24) $$B_2 \in \Re_+^n,\ C_2 = [0 \quad \cdots \quad 0 \quad 1] \in \Re_+^{1\times n}.$$

It is easy to check that

(25) $$\mathrm{rank}\begin{bmatrix} C_2 \\ C_2 A_2 \\ \vdots \\ C_2 A_2^{n-1} \end{bmatrix} = n \text{ if } a_k > 0,\ k = 1,\dots,n-1.$$

Theorem 8. Let p_1, p_2, …, p_n be the poles and z_1, z_2, …, z_{n-1} the zeros of the transfer function

(26) $$T_2(s) = C_2[I_n s - A_2]^{-1} B_2$$

If

(27) $$\mathrm{rank}[B_2 \quad A_2 B_2 \quad \cdots \quad A_2^{n-1} B_2] < n$$

then at least one zero of (26) is equal to its poles.

Proof. The proof is dual to the proof of Theorem 7.

Normal positive linear systems

Consider the transfer matrix of the form

(28a) $$T(s) = \frac{N(s)}{d(s)} \in \Re^{p\times m}(s),$$

where $N(s) \in \Re^{p\times m}[s]$ is the polynomial matrix and $d(s)$ is the least common denominator of the form

(28b) $$d(s) = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 .$$

Definition 6. The positive linear system with (28) is called normal if every nonzero second order minor of $N(s)$ is divisible (with zero remainder) by the polynomial $d(s)$.

The normal systems are insensitive to the change of their parameters [14].

Definition 7. The state matrix A of the linear system (1) is called cyclic if its minimal polynomial $\Psi(s)$ is equal to its characteristic polynomial

(29) $$\varphi(s) = \det[I_n s - A] .$$

The minimal polynomial $\psi(s)$ is related to its characteristic polynomial $\varphi(s)$ by [14]

(30) $$\psi(s) = \frac{\varphi(s)}{D_{n-1}(s)} ,$$

where $D_{n-1}(s)$ is the greatest common divisor of all $n-1$ order minors of the matrix $[I_n s - A]$.

Therefore, $\psi(s) = \varphi(s)$ if and only if $D_{n-1}(s) = 1$.

Theorem 9. The matrices A_1 and A_2 defined by (5) are cyclic.

Proof. By Definition 7 and (30) the matrices A_1 and A_2 are cyclic if and only if the greatest common divisor of all $n-1$ order minors of the matrices

(31)
$$[I_n s - A_1] = \begin{bmatrix} s+s_1 & -a_1 & 0 & \cdots & 0 & 0 \\ 0 & s+s_2 & -a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s+s_{n-1} & -a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & s+s_n \end{bmatrix},$$
$$[I_n s - A_2] = \begin{bmatrix} s+s_1 & 0 & 0 & \cdots & 0 & 0 \\ -a_1 & s+s_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & s+s_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & -a_{n-1} & s+s_n \end{bmatrix}$$

are $D_{n-1}(s) = 1$. It is easy to see that the minors corresponding to the first column and the n-th row of the matrix $[I_n s - A_1]$ and to the first row and the n-th column of the matrix $[I_n s - A_2]$ are equal to $a_1 a_2 ... a_{n-1}$. Therefore, $D_{n-1}(s) = 1$ and the matrices A_1 and A_2 are cyclic. □

Theorem 10. The positive linear system with the matrices A_1 and A_2 defined by (5) is normal for any $B \in \Re_+^{n\times m}$ and $C \in \Re_+^{p\times n}$.

Proof. By Definition 6 the positive linear system with A_1 (A_2) defined by (5) and any $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$ is normal if every nonzero second order minor of the matrix $N(s) = C[I_n s - A_1]_{ad} B$ is divisible by the polynomial $\det[I_n s - A_1]$.

Let $Z^{i_1 i_2 ... i_q}_{j_1 j_2 ... j_q}$ be the minor of the matrix Z with its i_1, i_2, ..., i_q rows and j_1, j_2, ..., j_q its columns. Then it is well-known [23] that the q-minor of the matrix $Z = PQ$ is given by

(32) $$Z^{i_1 i_2 ... i_q}_{j_1 j_2 ... j_q} = \sum_{1<k_1<...<k_q} P^{i_1 i_2 ... i_q}_{k_1 k_2 ... k_q} Q^{k_1 k_2 ... k_q}_{j_1 j_2 ... j_q} .$$

Note that the minors of the matrices B and C are independent of s. Using (32) for the matrix $C[I_n s - A_1]_{ad} B$ it is easy to see that its every nonzero second order minor is divisible by $\det[I_n s - A_1]$ since by Theorem 9 the matrix A_1 (A_2) is cyclic. Therefore, the positive linear system with A_1 (A_2) and any $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$ is normal. □

Example 2. Consider the positive linear system with the matrices

(33)
$$A_1 = \begin{bmatrix} -1 & a_1 & 0 \\ 0 & -2 & a_2 \\ 0 & 0 & -3 \end{bmatrix}, \; a_k > 0 \text{ for } k = 1,2,$$
$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \in \Re_+^{3\times 2},$$
$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \in \Re_+^{2\times 3}.$$

Taking into account that

(34)
$$d(s) = \det[I_3 s - A_1] = \begin{vmatrix} s+1 & -a_1 & 0 \\ 0 & s+2 & -a_2 \\ 0 & 0 & s+3 \end{vmatrix}$$
$$= (s+1)(s+2)(s+3) = s^3 + 6s^2 + 11s + 6$$

and

(35) $$[I_3 s - A_1]_{ad} = \begin{bmatrix} (s+2)(s+3) & a_1(s+3) & a_1 a_2 \\ 0 & (s+1)(s+3) & a_2(s+1) \\ 0 & 0 & (s+1)(s+2) \end{bmatrix}$$

we obtain

(36a)
$$N(s) = C[I_3 s - A_1]_{ad} B_1$$
$$= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$
$$\times \begin{bmatrix} (s+2)(s+3) & a_1(s+3) & a_1 a_2 \\ 0 & (s+1)(s+3) & a_2(s+1) \\ 0 & 0 & (s+1)(s+2) \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$
$$= \begin{bmatrix} n_{11}(s) & n_{12}(s) \\ n_{21}(s) & n_{22}(s) \end{bmatrix},$$

where

(36b)

$$
\begin{aligned}
n_{11}(s) &= b_{21}b_{31}a_2c_{12}(s+1)+b_{21}a_1c_{11}(s+3) \\
&\quad + b_{21}b_{31}c_{13}(s+1)(s+2)+b_{21}b_{11}c_{11}(s+2)(s+3) \\
&\quad + b_{12}c_{12}(s+1)(s+3)+b_{21}b_{31}a_1a_2c_{11}, \\
n_{12}(s) &= b_{22}b_{32}a_2c_{12}(s+1)+b_{22}a_1c_{11}(s+3) \\
&\quad + b_{22}b_{32}c_{13}(s+1)(s+2)+b_{22}b_{12}c_{11}(s+2)(s+3) \\
&\quad + b_{22}c_{12}(s+1)(s+3)+b_{22}b_{32}a_1a_2c_{11}, \\
n_{21}(s) &= b_{21}b_{31}a_2c_{22}(s+1)+b_{21}a_1c_{21}(s+3) \\
&\quad + b_{21}b_{31}c_{23}(s+1)(s+2)+b_{21}b_{11}c_{21}(s+2)(s+3) \\
&\quad + b_{21}c_{22}(s+1)(s+3)+b_{21}b_{31}a_1a_2c_{21}, \\
n_{22}(s) &= b_{22}b_{32}a_2c_{22}(s+1)+b_{22}a_1c_{21}(s+3) \\
&\quad + b_{22}b_{32}c_{23}(s+1)(s+2)+b_{22}b_{12}c_{21}(s+2)(s+3) \\
&\quad + b_{22}c_{22}(s+1)(s+3)+b_{22}b_{32}a_1a_2c_{21}.
\end{aligned}
$$

Therefore, the positive linear system with (33) is normal. Note that the matrices (5) for $a_k = 0$, $k = 1,...,n-1$ are equal and have the diagonal form

(37) $$A_d = \mathrm{diag}[-s_1 \quad -s_2 \quad \cdots \quad -s_n].$$

In this particular case Theorem 10 has the following form.

Theorem 11. The positive linear system with (37) and any $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$ is normal.

5. Normal positive linear electrical circuits

Consider linear electrical circuits composed of resistors, capacitors, coils and voltage (current) sources. As the state variables (the components of the state vector $x(t)$) we choose the voltages on the capacitors and the currents in the coils. Using Kirchhoff's laws we may describe the linear circuits in transient states by the state equations

(38a) $$\dot{x} = Ax + Bu,$$

(38b) $$y = Cx,$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$.

Definition 8. [26] The linear electrical circuit (38) is called (internally) positive if the state vector $x(t) \in \Re_+^n$ and output vector $y(t) \in \Re_+^p$, $t \ge 0$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \ge 0$.

Theorem 12. [26] The linear electrical circuit (38) is positive if and only if

(39) $$A \in M_n,\ B \in \Re_+^{n\times m},\ C \in \Re_+^{p\times n}.$$

The transfer matrix of the linear electrical circuit described by (38) can be always written in the form (28a).

Definition 9. The positive linear electrical circuit is called normal if every nonzero second order minor of $N(s)$ is divisible by $d(s)$.

Example 3. Consider the linear electrical circuit shown on Fig. 1 with given resistances R_k, inductances L_k, $k = 1,2,3$ and source voltages e_1, e_2.

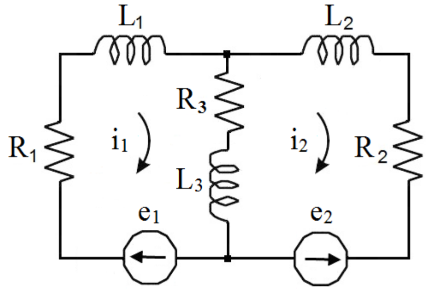

Fig. 1. Electrical circuit of Example 3

Using the mesh method for the electrical circuit we obtain

(40a) $$\begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix} \frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -R_{11} & R_{12} \\ R_{21} & -R_{22} \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where $i_1 = i_1(t)$, $i_2 = i_2(t)$ are the mesh currents and

(40b) $$\begin{aligned} &R_{11} = R_1 + R_3,\ R_{12} = R_{21} = R_3,\ R_{22} = R_2 + R_3, \\ &L_{11} = L_1 + L_3, L_{12} = L_{21} = L_3,\ L_{22} = L_2 + L_3. \end{aligned}$$

The inverse matrix

(41) $$\begin{aligned} L^{-1} &= \begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix}^{-1} \\ &= \frac{1}{L_1(L_2+L_3)+L_2L_3}\begin{bmatrix} L_{22} & L_{12} \\ L_{21} & L_{11} \end{bmatrix} \end{aligned}$$

has all positive entries.

From (40a) we obtain

(42a) $$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where

(42b) $$\begin{aligned} A &= L^{-1}\begin{bmatrix} -R_{11} & R_{12} \\ R_{21} & -R_{22} \end{bmatrix} \\ &= \frac{1}{L_1(L_2+L_3)+L_2L_3} \\ &\quad \times \begin{bmatrix} -L_2(R_1+R_3)-L_3R_1 & L_2R_3-L_3R_2 \\ L_1R_3-L_3R_1 & -L_1(R_2+R_3)-L_3R_2 \end{bmatrix}, \\ B &= L^{-1} \in \Re_+^{2\times 2}. \end{aligned}$$

Note that if

(43) $$L_1R_3 = L_3R_1 \text{ and } L_2R_3 - L_3R_2 > 0$$

then the matrix A has the form of the matrix A_1 defined by (5) and for

(44) $$L_2R_3 = L_3R_2 \text{ and } L_1R_3 > L_3R_1$$

the form of the matrix A_2. In both cases the electrical circuit is positive.

These considerations can be easily extended to n-mesh linear electrical circuits.

Following [26] let us consider the linear electrical circuit shown in Fig. 2 with given resistances R_k, $k = 1,...,8$, inductances L_2, L_4, L_6, L_8, capacitances C_1, C_3, C_5, C_7 and source voltages e_0, e_2, e_4, e_6, e_8.

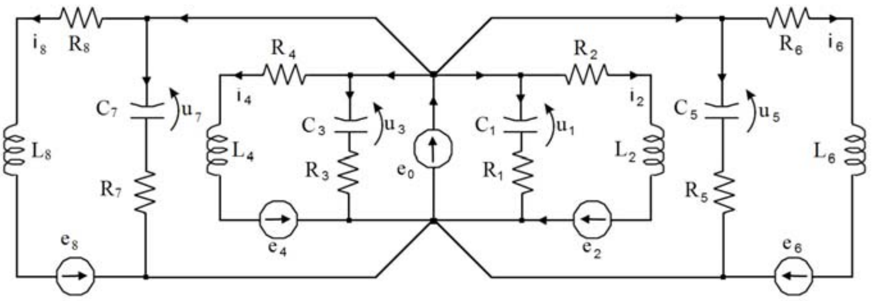

Fig. 2. Positive electrical circuit

Using Kirchhoff's laws we may write the equations

(45a) $$e_0 = u_k + R_k C_k \frac{du_k}{dt}, \; k = 1,3,5,7,$$

(45b) $$e_0 + e_j = R_j i_j + L_j \frac{di_j}{dt}, \; j = 2,4,6,8.$$

The equations can be written in the form

(46a) $$\frac{d}{dt}\begin{bmatrix} u \\ i \end{bmatrix} = A\begin{bmatrix} u \\ i \end{bmatrix} + Be,$$

where

(46b) $$u = \begin{bmatrix} u_1 \\ u_3 \\ u_5 \\ u_7 \end{bmatrix}, \; i = \begin{bmatrix} i_2 \\ i_4 \\ i_6 \\ i_8 \end{bmatrix}, \; e = \begin{bmatrix} e_0 \\ e_2 \\ e_4 \\ e_6 \\ e_8 \end{bmatrix}$$

and

$$A = \mathrm{diag}\begin{bmatrix} -\frac{1}{R_1C_1} & -\frac{1}{R_3C_3} & -\frac{1}{R_5C_5} & -\frac{1}{R_7C_7} & -\frac{R_2}{L_2} & -\frac{R_4}{L_4} & -\frac{R_6}{L_6} & -\frac{R_8}{L_8} \end{bmatrix},$$

(46c) $$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; B_1 = \begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 & 0 & 0 \\ \frac{1}{R_3C_3} & 0 & 0 & 0 & 0 \\ \frac{1}{R_5C_5} & 0 & 0 & 0 & 0 \\ \frac{1}{R_7C_7} & 0 & 0 & 0 & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} \frac{1}{L_2} & \frac{1}{L_2} & 0 & 0 & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} & 0 & 0 \\ \frac{1}{L_6} & 0 & 0 & \frac{1}{L_6} & 0 \\ \frac{1}{L_8} & 0 & 0 & 0 & \frac{1}{L_8} \end{bmatrix}.$$

The matrix $A \in M_8$ is diagonal and asymptotically stable and $B \in \Re_+^{8\times 5}$. Therefore, the electrical circuit is positive for any values of the resistances, inductances and capacitances and from Theorem 11 we have the following important theorem.

Theorem 13. Positive linear electrical circuit with diagonal matrix $A \in M_n$ and $B \in \Re_+^{n\times m}$, $C \in \Re_+^{p\times n}$ is normal for any values of the resistances, inductances and capacitances.

Concluding remarks

The notion of normal positive electrical circuit has been introduced and some specific properties of this class have been investigated. New state matrices of the positive linear systems and electrical circuits have been introduced and their properties have been analyzed (Theorems 7, 8, 9, 10 and 11). It has been shown that the positive electrical circuits with diagonal state matrices are normal for all values of their resistances, inductances and capacitances (Theorem 12). The considerations have been illustrated by numerical examples.

The considerations can be extended to fractional linear systems and electrical circuits.

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***Authors**: prof. dr hab. inż. Tadeusz Kaczorek, Politechnika Białostocka, Wydział Elektryczny, ul. Wiejska 45a, 15-351 Białystok, E-mail: kaczorek@ee.pw.edu.pl.*

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