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doi:10.15199/48.2019.11.18

Fast harmonics identification based on a compressive sensing approach

***Abstract**. The paper presents the application of a fast reconstruction algorithm, based on the theory of compressive sensing that can detect harmonics in an input signal. The problem of signal reconstruction is solved using a convex optimization by the linear programming algorithm. Additionally, to accelerate the convergence, a K-rank-order filter is applied in the signal's sparse domain. The numerical simulation carried out confirms the effectiveness of the algorithm used.*

Streszczenie.** W pracy przedstawiono implementację szybkiego algorytmu rekonstrukcji sygnału, opartego na teorii oszczędnego próbkowania, który może wykrywać harmoniczne w sygnale wejściowym. Zagadnienie rekonstrukcji sygnału jest problemem optymalizacyjnym rozwiązywanym za pomocą algorytmu programowania liniowego. Dodatkowo, aby przyspieszyć zbieżność rozwiązania zastosowano w rzadkiej dziedzinie sygnału filtr typu K-rank-order. Przeprowadzona symulacja numeryczna potwierdza skuteczność zastosowanego algorytmu. (**Szybka identyfikacja harmonicznych na podstawie oszczędnego próbkowana).

Keywords: harmonics identification, compressive sensing.
Słowa kluczowe: identyfikacja harmonicznych, oszczędne próbkowanie.

Introduction

Various new techniques for the identification and evaluation of harmonic sources in electricity supply systems are presented in the literature [1, 2, 3, 4].

In many practical applications, acquiring signals at ever higher sampling rates causes that large amounts of data must be stored in physical memory, which in turn leads to a huge memory occupancy. Thus, storing and processing or analysing the resulting large number of data has made it necessary to compress the desired information using the appropriate algorithms. Differently, from the typical approach usually implemented for signal compression, a new technique called compressive sensing (CS) uses a limited number of random linear projections to acquire efficient representations of compressible signals directly [5, 6, 7].

In general, the CS theory consists of three main issues. The first is determining the sparsest representation of a signal. Next, it is important to find an applicable compression transformation matrix, which approximates well the original N length signal for the least M coefficients. The last one concerns implementing proper reconstruction algorithm, which can recover original signal from observed M coefficients.

This paper presents the application of a fast reconstruction algorithm based on a convex optimization with the use of K-rank-order filter in the signal's sparse domain to accelerate the solution convergence. This filter has low computational complexity and is easy to implement. In this way, the computational burden associated with digital signal processing, which is performed in the next step of the measuring path, is reduced.

The paper is organized as follows. In the first section, the principles of the CS theory are given. Then, a sparse representation of a narrowband signal in the Fourier basis is proposed. The next section describes the random acquisition process. The algorithmic implementation of the reconstruction procedure is explained in the consecutive section. Then, numerical results are provided, which are followed by the conclusion in the last section.

Compressive sensing principles

Unlike traditional techniques, the innovative idea of the CS relies on the simultaneous execution of the acquisition and compression process. Time–domain sampling of discrete signals y can be expressed as [5]:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1N} \\ \vdots & \ddots & \vdots \\ \varphi_{M1} & \cdots & \varphi_{MN} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \varphi \cdot x \tag{1}$$

where: $y \in R^M$ – a measurement vector, $\varphi \in R^{MxN}$ – a measurement matrix, $x \in R^N$ - an input signal.

The measurement matrix φ allows to determine M ($M<<N$) of measured values from which a reliable reconstruction of the original signal is possible.

To apply the CS theory efficiently, two conditions must be met. The first concerns a sparsity of the input signal. Let x denotes an N-length real signal that can be expanded in an orthonormal basis as [5]:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} \psi_{11} & \cdots & \psi_{1N} \\ \vdots & \ddots & \vdots \\ \psi_{N1} & \cdots & \psi_{NN} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \psi \cdot a \tag{2}$$

where: $\psi \in R^{NxN}$ – a transformation matrix, $a \in R^N$ – a sparse representation of the signal of x in the matrix ψ.

The signal is sparse, which means it consists of only a small number of basic functions. The second condition states that the measurement matrix must be incoherent with basis in which signal has sparse representation.

If the input signal is K-sparse on basis ψ as well as matrix φ and ψ are incoherent, the signal x can be reconstructed using limited number of linear projections ($K<M<<N$).

By inserting (2) into (1) the measurement signal y becomes:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1N} \\ \vdots & \ddots & \vdots \\ \varphi_{M1} & \cdots & \varphi_{MN} \end{bmatrix} \cdot \begin{bmatrix} \psi_{11} & \dots & \psi_{1N} \\ \vdots & \ddots & \vdots \\ \psi_{M1} & \dots & \psi_{M1} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} \tag{3}$$

and

$$(4)\quad \Theta = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1N} \\ \vdots & \ddots & \vdots \\ \varphi_{M1} & \cdots & \varphi_{MN} \end{bmatrix} \cdot \begin{bmatrix} \psi_{11} & \dots & \psi_{1N} \\ \vdots & \ddots & \vdots \\ \psi_{M1} & \dots & \psi_{M1} \end{bmatrix}$$

where: $\Theta \in R^{MxN}$ – a reconstruction matrix.

In order to reconstruct the sparse signal x, the equation (3) has to be resolved, which gives infinite number of possible solutions. Therefore, optimization algorithms based on the l_1-norm minimization is usually applied [5]:

$$(5)\qquad \hat{a} = \arg\min \|a\|_1 \quad subject\ to\ \ y = \Theta \cdot a$$

where: $\hat{a}$ - the estimate of a and $\|a\|_1$ denotes l_1-norm of a.

Complying with (5), the input signal estimate can be found as [8]:

$$(6)\qquad \hat{x} = \varphi \cdot (\Theta^T \cdot \Theta)^{-1} \cdot \Theta^T \cdot y$$

where: $(matrix)^{-1}$ - the pseudoinverse matrix of $matrix$.

Sparse representation of the input signal

It is assumed, that the input signal is a narrowband signal that can be described as a superposition of K sinusoids [9]:

$$(7)\qquad x_n = \frac{1}{N}\sum_{k=1}^{K} X_k \cdot \exp(j \cdot \frac{2\pi}{N} \cdot n \cdot k)$$

where: X_k – the DFT of signal x_n.

Such signal has a K-sparse representation in terms of the discrete Fourier transform (DFT), since:

$$(8)\qquad X_k = \sum_{n=1}^{N} x_n \cdot \exp(-j \cdot \frac{2\pi}{N} \cdot n \cdot k)$$

The transformation matrix ψ , constructed on the Fourier basis is described by [2]:

$$(9)\qquad \psi_{n,k} = \frac{1}{\sqrt{N}} \cdot \exp(j \cdot \frac{2\pi}{N} \cdot n \cdot k)$$

The matrix ψ is a fixed signal feature; no degree of freedom is possible.

Random sampling

CS is mainly concerned with low coherence between the transformation matrix ψ and the measurement matrix φ. In the literature, some frequent examples of incoherent basis couples can be found [5]. Random matrices are largely incoherent with any fixed basis ψ [5].

If the random measurement matrix φ is created based on the Bernoulli distribution, with the ones probability p [10], and the exemplary element of y is given by:

$$(10)\qquad y_i = [0\ 0\ 0 \cdots 1\ 0\ 1] \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix},$$

what can be generalized as:

$$(11)\qquad y = \langle \varphi, x \rangle = \sum_{j=0}^{N-1} \varphi_{i,j} \cdot x_j \ ,$$

where: $\varphi_{i,j}$ - the *(i, j)*th entry of the random binary matrix φ.,

Signal reconstruction

The problem of signal reconstruction is resolved based on a convex optimization by the linear programming algorithm [7]. Additionally, to accelerate the convergence, a K-rank-order filter is applied in the signal's sparse domain. The reconstruction formula is described as follows [11]:

$$(12)\qquad \overline{x} = \frac{1}{1+i}\sum_{j=0}^{i-1} \frac{1}{p_j(1-p_j)} y_j \cdot \varphi_j$$

where p_j is the ones probability at the j^{th} iteration.

The algorithm operates in a loop, and in each iteration, it verifies whether equation (12) actually converges.

The processing loop terminates when the threshold condition *t* meets [11]:

$$(13)\qquad 0 \ll t \le \frac{|y_i|}{|y_i| + |y_i - \langle \varphi_i, \hat{x}_i \rangle|} \le 1$$

where: $\hat{x}_i$ – the estimate of x at the i^{th} iteration.

The Fourier transformation T is applied to sparsify the solution. The estimate of x_i is expressed as:

$$(14)\qquad \hat{x}_i = T^{-1}\left\{Rank\left(T\left\{\overline{x}\right\}\right)\right\}$$

where: $Rank$ (*)– denotes a K-rank-order filter.

Let's consider a vector containing samples $X_k = T\{\overline{x}\} = [X_1, \cdots, X_N]$. The filter extracts the K most significant components from the input vector and assigns zeros to the remaining places (see Fig. 1). This way, the computational burden of the inverse Fourier transform, performed in the next step, is reduced.

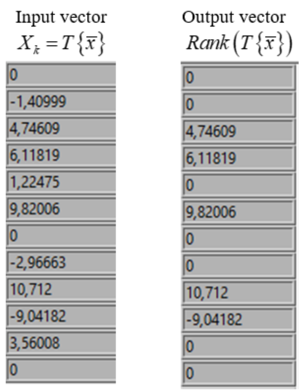

Fig. 1. The exemplary results of the K-rank-order filter operation with K equal to 5

Numerical simulation

The simulations were performed using a program designed based on an accessible application in the LabVIEW environment [11]. As an example, a multi-tone signal with fundamental harmonic 50 Hz has been generated, according to the parameter sets shown in Tab. 1. The sampling frequency is equal to 10 kHz and the length of the time window is equal to 1000 samples. The time waveform and sparse representation of the tested signal in the Fourier domain is presented in Fig. 2.

Table 1. The parameters of the input signals

Harmonic no	Amplitude [-]			
	set no 1	set no 2	set no 3	set no 4
1	1	10	100	100
3	1	6	5	10
5	1	4	3	10
7	1	2	1	10

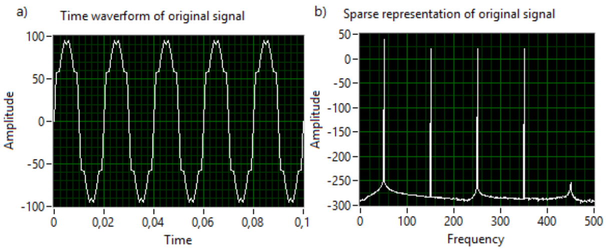

Fig. 2. The segment of time waveform (a) and normalized DFT components (b) of input signal (set no 4)

In the case of generating the tones with the same level of amplitude (set no 1), the algorithm identifies harmonics with accuracy above 97%. The most accurate signal reconstruction (99,32%) is obtained for 92 sampled measurement signals, acquired with the ones probability p equals 0,3 (see Fig. 3). For the input signal with slightly different levels of amplitude (set no 2), the reconstruction based on 102 measurements and the same features of the measurement matrix does not allow the correct detection of all harmonics (see Fig 4). The proper reconstruction requires three times more measurements, i.e. about 350. The scenario with the dominant fundamental harmonic (set no 3) shows that 500 sampled measurements are not sufficient to identify all components in the frequency domain (see Fig. 5). The application of the presented reconstruction algorithm to the signal in which the dominant harmonic occurs requires the pre-conditioning of the signal. It consists in the separation of the fundamental component from the examined waveform. The increasing of the amplitude level of the higher harmonics (set no 4) results in the possibility of harmonic identification (see Fig. 6). In this case, the number of random samples reaches 800.

All FFT analyses were carried out with a 10 Hz resolution.

The sensitivity of the Fourier amplitude detection depends on the conditions of acquisition of the input signal, i.e. the resolution of the AD converter.

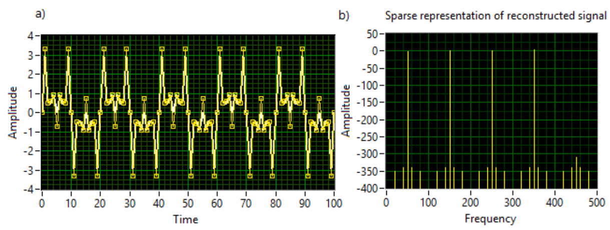

Fig. 3. The segments of time waveforms of the original (white line) and reconstructed signal (dotted yellow line) (a) and normalized DFT components of the reconstructed signal (b). The set no 1

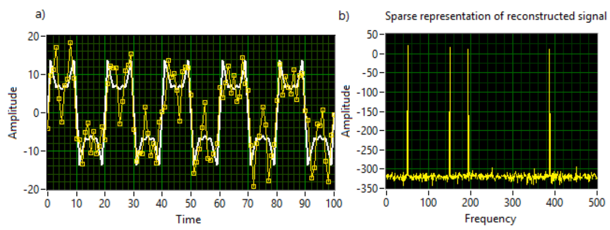

Fig. 4. The segments of time waveforms of the original (white line) and reconstructed signal (dotted yellow line) (a) and normalized DFT components of the reconstructed signal (b). The set no 2

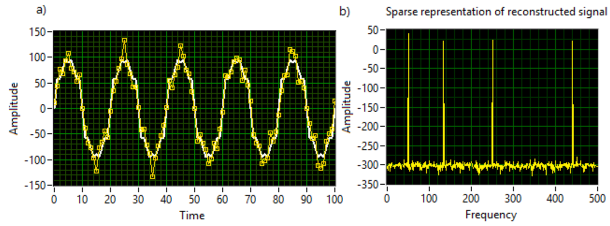

Fig. 5. The segments of time waveforms of the original (white line) and reconstructed signal (dotted yellow line) (a) and normalized DFT components of the reconstructed signal (b). The set no 3

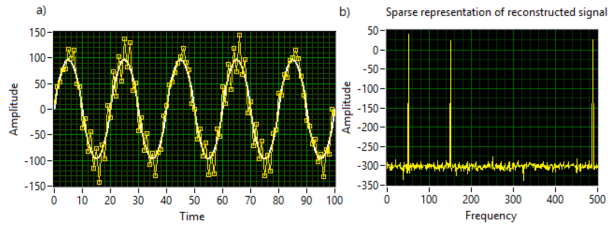

Fig. 6. The segments of time waveforms of the original (white line) and reconstructed signal (dotted yellow line) (a) and normalized DFT components of the reconstructed signal (b). The set no 4

Conclusion

The preliminary results of numerical simulations performed, using the fast reconstruction algorithm, show the limitations of effective reconstruction based on the CS.

The good reconstruction accuracy occurs in the case of multi-tone signal consisting of components whose amplitude levels do not differ significantly.

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