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doi:10.15199/48.2019.04.36

Criterion for transient behaviour in a nonlinear Duffing oscillator

***Abstract**. The paper proposes a criterion for determining transient behaviour in a nonlinear Duffing oscillator. For this purpose studies of specific attractors typical of the system have been conducted. Exactly defined deviation value of ∆ with respect to the mean value of the surface areas bounded by the successive trajectory cycles has been assumed as the termination of the transient behaviour.*

Streszczenie.** W pracy zaproponowano kryterium wyznaczania czasu trwania procesu przejściowego w nieliniowym oscylatorze Duffinga. W tym celu badano specyficzne atraktory charakteryzujące ten układ. Za kryterium końca procesu przejściowego przyjęto ściśle zdefiniowaną wartość odchyłki ∆ od wartości średniej pól powierzchni ograniczonych kolejnymi cyklami trajektorii. **Kryterium wyznaczania czasu trwania procesu przejściowego w nieliniowym oscylatorze Duffinga

Keywords: transient trajectory, transient behavior, Duffing oscillator, nonlinear oscillations, criterion of transient behaviour.
Słowa kluczowe: trajektoria przejściowa, proces przejściowy, oscylator Duffinga, drgania nieliniowe, kryterium stanu przejściowego.

Introduction

From a practical point of view, to determine the duration of transient behavior in nonlinear physical systems constitutes both an important and interesting problem. Transient behaviour occurs both in mechanical systems as well as electrical and electronic ones. In stable linear systems, it is assumed that the transient process fades after a time equal to five time constants. For an autonomous system analysis a more exact assessment of the duration of transients was presented in [1]. It used a Lyapunov function to define time t_{tr}, in which the trajectory of an autonomous system, starting from the initial state, reaches a specific area including the origin of the coordinate system. In that case, $t_{tr} \leq -\vartheta \ln V[\mathbf{x}(t_{tr})]/V[\mathbf{x}(0)]$, where $V[\mathbf{x}]$ is a Lyapunov function, while ϑ is the largest eigenvalue of a matrix determined from dependency $\vartheta = \max_{x}(V(x)/-\dot{V}(x))$.

The analysis of transient behaviour in nonlinear systems is more complicated as the superposition principle cannot be used here and, as a result, transient and steady components cannot be separated. In harmonic enforcement, nonlinear systems are characterized by the occurrence of a non-sinusoidal response. In many cases, they are systems with chaotic dynamics in which transient chaos are distinguished [2, 11]. The duration of the transient behaviour depends, among others, on the choice of initial conditions and the values of the system parameters. A great number of physical phenomena are modeled by basic differential equations. For example, vibrations in electrical, electronic and mechanical systems can be analyzed using the Duffing equation [3, 4]. The system of three Lorenz equations has been used to characterize the convective movement occurring, for instance, in the Earth's atmosphere [5, 6]. Electronic oscillators with non-linear damping are described by Van der Pol's equation [7, 8].

Below, we present a brief overview of the works on transient behavior analysis of nonlinear systems. Paper [8] presents an analysis of control systems dynamics described by the nonlinear Duffing and Van der Pol equations with chaotic enforcement. The authors showed that the proposed controller was found to be highly efficient in system tracking and had swift response time.

Considerable attention was paid to the analysis of transient behaviour in systems described by the Lorenz equation. For example, paper [9] provides a method called *partial control* that makes it possible to keep the analyzed system in transients and prevent the transient trajectories from escaping to the external attractor. On the other hand, paper [10] presents an analysis of transients dependent on changes in the initial conditions in the Lorenz model of atmospheric phenomena.

In paper [11] the authors described long chaotic transients in complex networks of pulse-coupled oscillators. It was shown that small changes in the structure of a complex system have a decisive influence on its dynamics.

A practical application of the assessment of the effect of transients on the operation of a physical system is presented in [12]. The paper analyzes specific transient basins in a gearbox where unwanted rattling occurs during chaotic vibrations.

In this work, which analyzes a non-linear system, we are considering a stable system with periodic dynamics. We investigate the transient trajectories with the set initial conditions and assumed parameters which are convergent to a repetitive loop (Fig. 2). The time interval in which this specific state of the system occurs is defined as transient behaviour (transients) and denoted by t_{tr}.

As mentioned above, an example of nonlinear dynamics are physical systems described by the Duffing equation. A general form of this equation is as follows:

(1) $$d^2z/dt^2 + a_1 dz/dt + a_2 z + a_3 z^3 = b_1 \cos(b_2 t + b_3)$$

where: a_1, a_2,a_3 are the parameters of the analyzed system, while b_1, b_2, $b_3$are the driving force parameters.

In mechanical systems, coefficients a_2, a_3 characterize a nonlinear spring, while in electric systems they describe approximation of the magnetization curve. In equation (1) *z(t)* stands for displacement and *ż(t)* for speed. Coefficient a_1 is related to the damping phenomenon in the considered system. After the transformation of equation (1) into a system of two equations, it is usually analyzed on the phase plane.

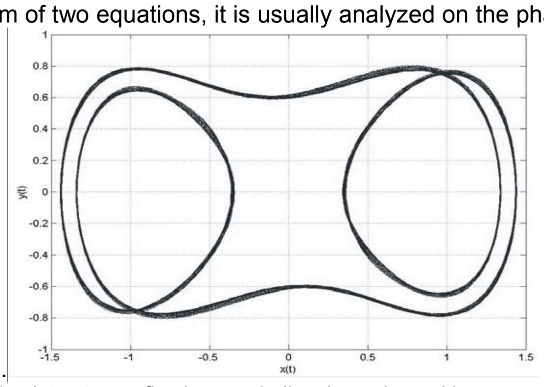

Fig. 1. Attractor reflecting periodic dynamics without marked transient trajectory and initial conditions [14].

It can be stated that in the periodic dynamics, the trajectories are found in a repetitive loop (Fig. 2). Very often, these trajectories are presented in the scientific literature without indicating the assumed initial conditions and the resulting transient trajectory [13, 14] (Fig. 1).

Considering the above mentioned remarks, the main goal of this work is to develop a criterion to define the duration of the transient behaviour (time t_{tr}). For this purpose a model of Duffing oscillator is used.

The proposed criterion for determining the transient behaviour

Equation (1) can be represented by the system of two first order equations. To this end, assuming $z(t)= x_1(t)$, and $\dot{z}(t) = x_2(t)$, we obtain:

$$\dot{x}_1 = x_2$$

(2) $$\dot{x}_2 = -a_1x_2 - a_2x_1 - a_3x_1^3 + b_1cos(b_2t+b_3)$$

In our further studies state variables $x_1(t)$ and $x_2(t)$, are assumed to be dimensionless. For b_2 =1, we obtain a driving force period of 2π. Adopting the above assumption and assuming a_1=0.5, we obtain an attractor reflecting the periodic oscillations in the system (Fig. 2), whereas waveform evolutions are illustrated in Fig. 3.

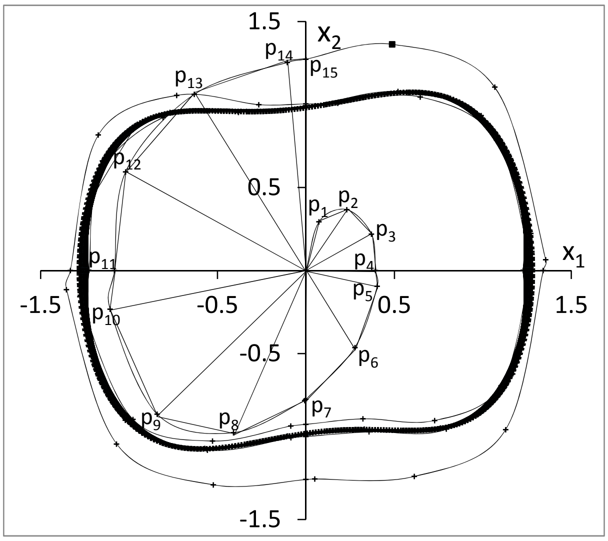

Fig. 2. Attractor reflecting periodic dynamics of the system described by equation (2) with transient trajectory. $x_1(0)=0$, $x_2(0)=0$, $a_1=0.5$, $a_2=0.27$, $a_3=1.2$, $b_1=1$, $b_2=1$, b3=0.5.

The lines shown in Fig. 2 describe the transient trajectory that is convergent to the repetitive loop that reflects periodic dynamics. As mentioned above, the basic goal of this paper is to develop a criterion that would make it possible to define the time interval in which the transient behavior t_{tr} occurs in the system. In our considerations zero initial conditions $x_1(0)=0$ and $x_2(0)=0$ have been taken into account. They correspond to point $p_0(0,0)$ on the phase plane (Fig. 2) and time t_i. The end of one complete cycle in transients is represented by point $p_{15}(x_1=0, x_2=1.273)$.

The first complete cycle time is 5.48s<2π. In the state of periodic dynamics, the phase trajectory enters the repetitive loop and period T (full cycle time) for the adopted parameter b_2=1 equals 2π. The individual points of the trajectory are the vertices of the triangles marked in Fig. 2 as p_1, p_2, p_3, …, N, where N-1 is the last triangle of the full first cycle in the analysis of the time interval. The equations (2) are solved by one of the numerical methods assuming the integration step h ensuring the stability of the method [15]. The sum of the areas of the triangles for one full cycle, eg from point $p_1(0.074, 0.294)$ to point $p_{15}(0, 1.273)$ is denoted as $Pc_1=(\sum_{n=1}^{N-1} S_n)_1$, $N=15$. For the second cycle from point $p_{15}(0, 1.273)$ the sum is denoted by $Pc_2=(\sum_{n=1}^{M} S_n)_2$, where N-1 and M = 16 are the number of triangle areas in the first and second cycle, respectively. Generally:

(3) $$Pc_i = (\sum_{n=1}^{L} S_n)_i$$

where i = 1,2,…,K is a consecutive cycle and L the number of triangles in a given cycle (Fig. 2).

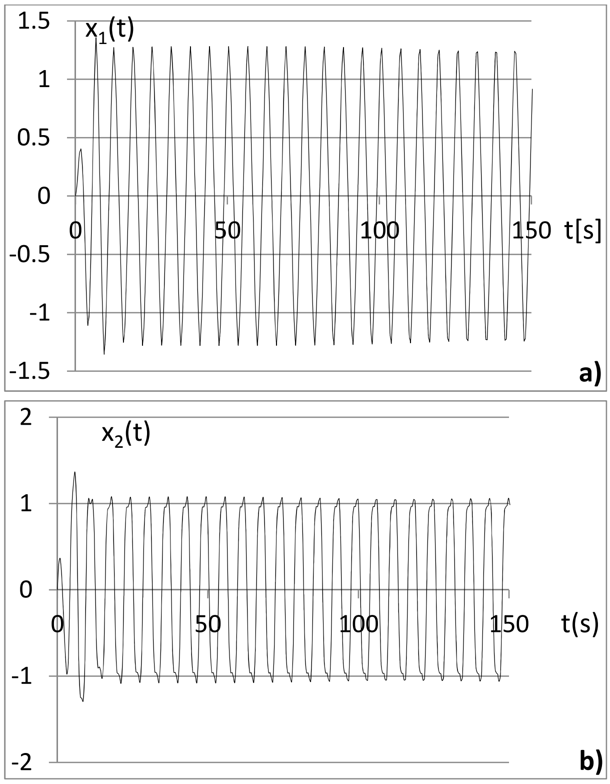

Fig. 3. Waveform evolutions of the Duffing oscillator (a_1=0.5), a) $x_1(t)$, b) $x_2(t)$

In the transient behaviour Pc_i values differ significantly from average P_K value (Fig. 4). In the time t_{tr} the trajectory enters in a repetitive loop (Fig. 2). We assume that time t_{tr} after which the transient trajectory enters periodic dynamics with the assumed difference Δ from the average value computed from the end of the time of the analysis, defines the end of the transient behaviour in the system (Fig. 4).

If we assume 'i' to denote the subsequent cycle number of the integration time (2), then the average value calculated from the end of the time of analysis equals to:

(5) $$P_{K_i} = \frac{P_{C_K} + P_{C_{K-1}} + ... + P_{C_i}}{K - i - 1}$$

Therefore, the periodic dynamics occurs for:

(6) $$\frac{\left|P_{C_i} - P_{K_i}\right|}{P_{K_i}} \geq \Delta$$

where i is the subsequent cycle from $t_i=0$. In this paper $\Delta=5\cdot10^{-3}$ is assumed.

Taking into account the above considerations, we present the definition of the transient behaviour in the systems described by the Duffing equation.

Definition. *In the system described by equations $\dot{x}=f(x(t),u(t),t)$ where $x(t)\in R^n$ and $u(t)\in R^m$, the transient behavior t_{tr} is called time t, in which the transient trajectory enter attractor loop with condition* $\frac{\left|P_{C_i} - P_{K_i}\right|}{P_{K_i}} \geq \Delta$, *where i is the subsequent cycle starting from t_i, and Δ is a sufficiently small number.*

The algorithm for determining the transient behaviour in these systems and its procedure is given below:

Step 1: Specify the integration step h of the system of nonlinear equations, perform the approximation of the surface inside the trajectory using the sum of the area of the triangles (Fig. 2).

Step 2: Calculate the mean value of the areas of the subsequent cycles, computing the mean value from the end of the integration time t_e of the equations according to formula (5).

Step 3: Determine the duration of the transient behaviour from dependence (6).

Step 4:Create charts to illustrate the Pc_i values (3) and average values of successive cycles of the attractor with time t_{tr}.

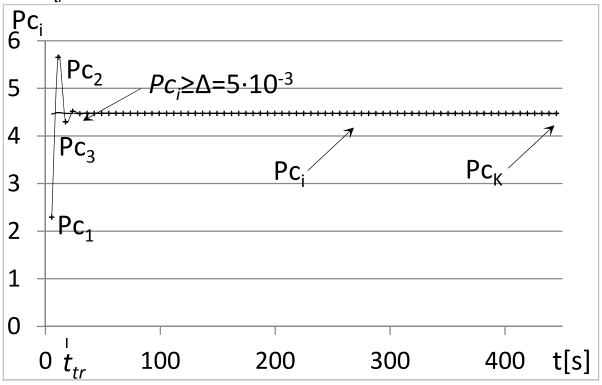

Fig. 4. Plot of Pc_i values (3) and average values of successive cycles of the attractor with time t_{tr} (a_1=0.5, a_2=0.27, a_3=1.2, b_1=1.0, b_2=1.0, b_3=0.5).

Study of the impact of system parameters on the transient behaviour

The transient behaviour of the Duffing oscillator (1) depends on the parameters that occur both on the left and right side of the equation. Parameters a_1, a_2, a_3 are characterized by: damping a_1, nonlinear spring a_2, a_3, i.e. own parameters of the system. The right side of the equation are the driving force parameters. The following values have been assumed for the study of system behavior:

$$\begin{aligned} &a_1 = 0.045 \div 0.5, && b_1 = 0.5, \\ &a_2 = 0.27, && b_2 = 1, \\ &a_3 = 1.2, && b_3 = 0.5, \\ &x_1(0) = 0,\ x_2(0) = 0. && \end{aligned} \tag{7}$$

The values of the coefficients have been chosen from the literature review on the topic [3]. Parameter a_1 was changed within the range of 0.045 to 0.5. Fig. 5 shows the dependence of time t_{tr} on value a_1.The other parameters were assumed to be constant. Time t_{tr} increases with decreasing a_1 values, which is consistent with the principle of the dependence of the transients on the damping values in the system.

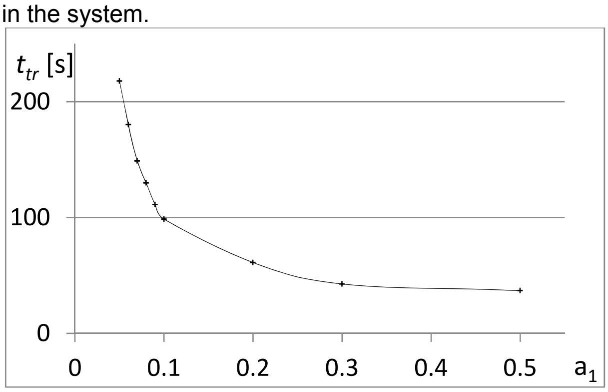

Fig. 5. Dependence of transient behaviour (time t_{tr}), on the value of a_1.

Examples of attractors and determination of transients t_{tr} in the system are shown in Figs. 6, 7, 8 and 9.

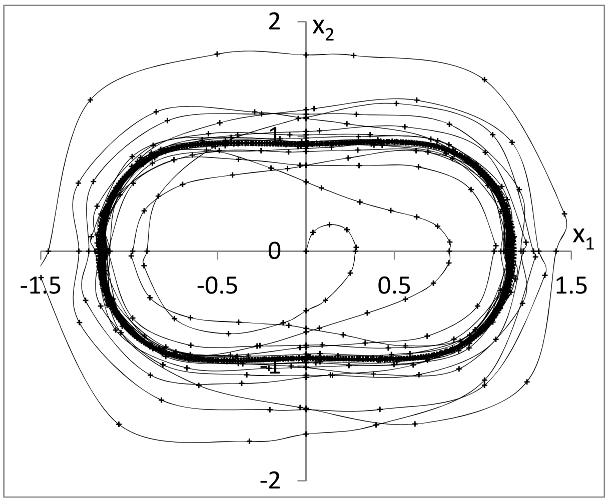

Fig. 6. Attractor illustrating the periodic dynamics of the system described by (2) for a long duration of transients: a_1=0.07 and the remaining parameters as given in (7).

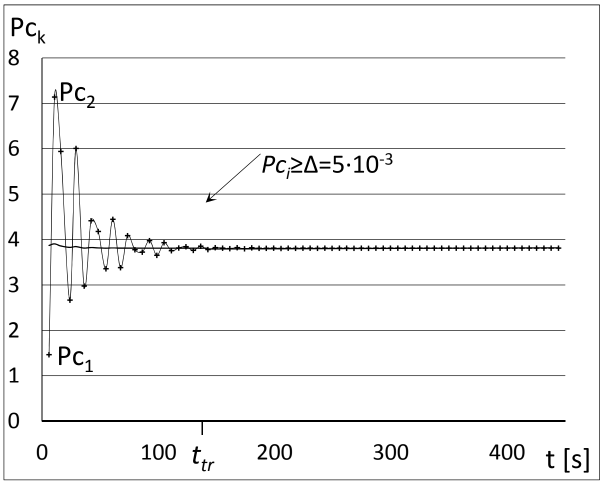

Fig. 7. Determination of time t_{tr} with a small value a_1=0.07.

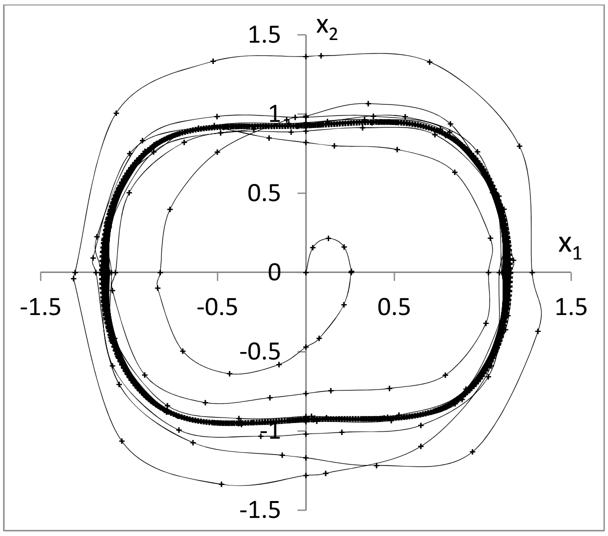

Fig. 8. Attractor reflecting the periodic dynamics of the system with a higher damping value: a_1=0.2 and the remaining parameters as given in (7).

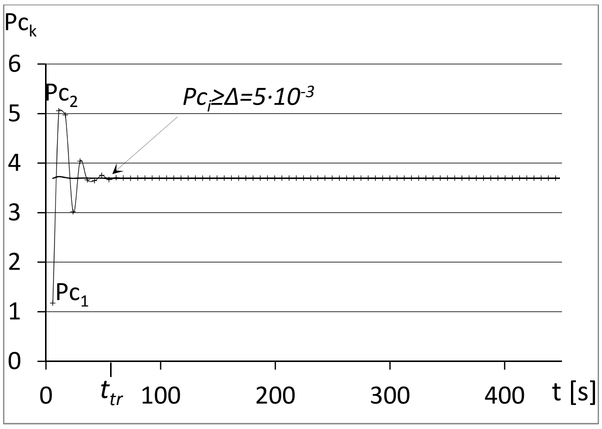

Fig. 9. Determination of time t_{tr} with a_1=0.2.

An equivalent measure of the transient behaviour in the system is the following expression:

(8) $$E=\int_0^{t_{tr}} x_1^2 dt+\int_0^{t_{tr}} x_2^2 dt$$

Since x_1 is, for example, a displacement, and x_2 speed, formula (8) represents the total energy found in the system described by (2) [16]. A dependency plot E=f(a_1) is shown in Fig. 10. This plot is qualitatively consistent with the one shown in Fig. 5.

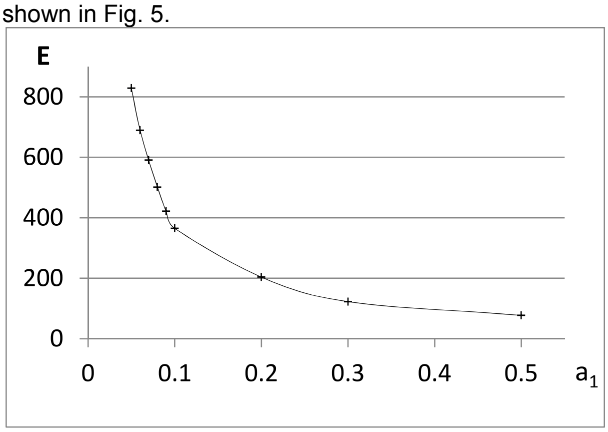

Fig. 10. Dependence of system energy on value of a_1.

Final remarks and conclusions

The paper proposes an original criterion for determining the transient behaviour in nonlinear systems. The criterion is illustrated using the Duffing oscilator that models both mechanical as well as electrical systems. The measure of the transient behaviour is time t_{tr} defined using the deviation *Δ* from the mean value expressed by formula (6) and computed from the end to the beginning of the time interval t_e -t_i (backward analysis). Time t_e is the final value of the time analysis, whereas t_i stands for the initial time value. The time t_{tr} at which the deviations should not be less than the assumed value, in this case $\Delta=5\cdot10^3$, is the end of the transient process in the system.

A separate issue is the assessment of the transient behaviour using total energy *E* present in the system at known time t_{tr} [16]. The energy defined by formula (8) is associated with the Lyapunov function of $V(x)=x_1^2+x_2^2$ [1], in which $\sqrt{x_1^2+x_2^2}$ is the length of segment *0P*$_n$, where P_n is an example trajectory point on the phase plane (Fig. 2).

The paper shows that trajectories analyzed on the phase plane comprise a considerable amount of data concerning the transient behavior in the Duffing equation.

The computations were made using a proprietary program written in C++. In order to solve equation (2) the Euler's method with the integration step h = 1.0e^{-6} was applied [15]. The program simultaneously calculates all the other elements of the presented analysis. Some selected results were compared with the computations obtained using Mathematica software [17].

The proposed criterion can be applied to Van der Pol, Lorenz and nonlinear Chua equations described, for example in papers [9, 13]. In the last two systems the transients can be determined in a three-dimensional phase space, with a larger number of state variables in R^n space.

Nonlinear systems are characterized by a high impact of initial conditions on the results of the analysis. The dependence of the transient behaviour on the initial conditions using the original criterion will be the subject of another article.

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