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Fourth Order Qualitative Inspection of Nonlinear Neutral Delay Difference Equation

***Abstract**. This paper aim is to investigate the nonlinear neutral fourth order difference equation in the form $\Delta^3\left(a_e\Delta\left(w_e - p_e w_{e-\alpha}\right)\right) + q_e f(w_{e-\beta}) = 0$. We establish some conditions to assure that all solutions to this equation are oscillatory or non-oscillatory. We derived this using summation averaging technique and comparison principle. The main outcomes are illustrated using examples.*

Streszczenie. W artykule badano nieliniowe równanie różnicowe czwartego rzędu : $\Delta^3\left(a_e\Delta\left(w_e - p_e w_{e-\alpha}\right)\right) + q_e f(w_{e-\beta}) = 0$. Ustalamy pewne warunki, aby zapewnić, że wszystkie rozwiązania tego równania są oscylacyjne lub nieoscylacyjne. Wyprowadziliśmy to za pomocą techniki uśredniania sumowania i zasady porównania. Główne wyniki zilustrowano na przykładach. (Badane nieliniowego równania różnicowe czwartego rzędu)

Keywords: Oscillation; Nonlinear; Neutral Difference Equation.
Słowa kluczowe: oscylacje, równanie różnicowe

Introduction

In the present paper, we have obtained some criteria for nonlinear neutral fourth order difference equation

$$(1) \qquad \Delta^3\left(a_e\Delta\left(w_e - p_e w_{e-\alpha}\right)\right) + q_e f(w_{e-\beta}) = 0, e \geq N$$

One or more of the following assumptions will be applied throughout the rest of our investigations:

(H_1) $\{a_e\}, \{p_e\}, \{q_e\}$ are positive real sequences and $q_e \neq 0$ for all the value of e, where $N = \{e_1, e_2, e_3, e_4 .....\}$

(H_2) α, β are positive integer and f is a real valued continuous function.

(H_3) $a_{e+3}\Delta k_{e+3} + 3a_{e+1}\Delta k_{e+1} > 3a_{e+2}\Delta k_{e+2}$

(H_4) $f(u+v) = f(u) + f(v)$

(H_5) $\dfrac{f(u)}{u} = \rho > 0$

(H_6) $v_n = \sum_{n=e_0}^{e} \dfrac{1}{a_n}\rho q_n k_{n-\beta}(1 + p_{n-\beta}) < \infty$

(H_7) $\sum_{n=e_1}^{e-1} \dfrac{1}{a_n} \to \infty$ as $e \to \infty$

(H_8) $f(u) = u$ for all $u \neq 0$

Equation (1) satisfy the real sequence $\{w_e\}$ for all $e \in N$. The solution of equation (1) is said to be oscillatory if it is neither eventually positive nor eventually negative and it is non oscillatory otherwise. The study of oscillatory solutions of neutral delay difference equations has gotten a lot of attention in recent years. Higher-order equations, particularly third and fourth order equations e.g. (2-11) have gotten far less attention.

In [2], the authors considered with the oscillatory property of neutral fourth order delay difference equation in the form

$$\Delta\left(c_n\Delta^2\left(a_n\Delta\left(y_n + b_n y_{n-\tau}\right)\right)\right) + q_n f\left(y_{n-\sigma}\right) = 0$$

are established.

In [3], the authors considered the oscillation property of nonlinear third order difference equation in the form

$$\Delta\left(\frac{1}{r_{n-1}}\Delta\left(\frac{1}{a_{n-1}}\left(\Delta y_{n-1}\right)^{\alpha}\right)\right) - \frac{1}{p_{n-1}}\left(\Delta y_{n-1}\right)^{\alpha} + \frac{1}{q_n} f\left(y_n\right) = 0$$

are established.

In [4], the authors studied the oscillatory behaviour of fourth order linear and nonlinear difference equation of the form $\Delta^3\left(\dfrac{a_n}{q_n}\Delta y_n\right) + q_n y_{n+1} = 0$ and $\Delta^3\left(\dfrac{a_n}{q_n}\Delta y_n\right) + q_n f\left(y_{n+1}\right) = 0$ are obtained.

In [6], the authors studied some sufficient conditions for the oscillation of certain fourth order nonlinear difference equation of the form $\Delta^2\left(\dfrac{1}{p_n}\Delta^2 y_n\right) + q_n f\left(y_{n-\tau_n}\right) = 0$ are obtained.

In [7], the authors considered the oscillation of nonlinear difference equation in the form $\Delta^2\left(\dfrac{1}{a_n}\Delta^2\left(y_n + p_n y_{n-k}\right)\right) + q_n f\left(y_{\sigma(n)}\right) = 0$ are established.

In [9], the authors studied the linear neutral delay difference equation in the form

$$\Delta\left(a(n)\Delta^2\left(x(n) + p(n)x(\tau(n))\right)\right) + q(n)x(\sigma(n)) = 0$$

are established.

In [10], the authors studied the oscillation of third order neutral delay difference equation $\Delta^3\left(y_n + p_n y_{n-k}\right) + q_n f\left(y_{n-l}\right) = 0$ are obtained.

The goal of this study is to find some new adequate condition for equation (1) of fourth order nonlinear difference equation. The findings in this research were influenced by those found in [7, 10, 11].

Oscillatory results

We have got some sufficient conditions for the oscillation of equation (1) in this section.

Theorem 1 Assume $\sum_{s=e_0}^{e}\frac{\gamma_s}{a_s}f(1+p_{s-\beta}w_{s-\alpha-\beta}=\infty$, and if the condition (H_3), (H_4) holds then every solution of (1) is oscillatory.

Proof: Suppose $\{w_e\}$ is an non oscillatory solution of (1). Without loss of generality, assume w_e is eventually positive.

$$\Delta^3(a_e\Delta k_e)+q_ef(w_{e-\beta})=0$$

$$\Delta^3(a_{e+1}\Delta k_{e+1}-a_e\Delta k_e)+q_ef(k_{e-\beta}+p_ew_{e-\alpha-\beta})=0$$

(2) $a_{e+3}\Delta k_{e+3}-3a_{e+2}\Delta k_{e+2}+3a_{e+1}\Delta k_{e+1}-a_e\Delta k_e+q_ef(k_{e-\beta}+p_{e-\beta}w_{e-\alpha-\beta})=0$

$$a_{e+3}\Delta k_{e+3}+3a_{e+1}\Delta k_{e+1}+q_ef(k_{e-\beta}+p_{e-\beta}w_{e-\alpha-\beta})=3a_{e+2}\Delta k_{e+2}+a_e\Delta k_e$$

$$q_ef(k_{e-\beta}+p_{e-\beta}w_{e-\alpha-\beta})>a_e\Delta k_e$$

$$q_ef(k_{e-\beta}+p_{e-\beta}w_{e-\alpha-\beta})-a_e\Delta k_e=0$$

Let $x_e=\frac{\gamma_e\Delta k_e}{q_ef(k_{e-\beta})}$

$$q_ef(k_{e-\beta}+p_{e-\beta}w_{e-\alpha-\beta})-\frac{a_eq_ef(k_{e-\beta})x_e}{\gamma_e}=0$$

$$\frac{q_ef(k_{e-\beta}+p_{e-\beta}w_{e-\alpha-\beta})}{a_eq_ef(k_{e-\beta})}>x_e$$

$$\frac{f(1+p_{e-\beta}w_{e-\alpha-\beta})\gamma_e}{a_e}>x_e$$

$$\frac{\gamma_e}{a_e}f(1+p_{e-\beta}w_{e-\propto-\beta})>x_e$$

Generalizing the above inequality

$$\sum_{s=e_0}^{e}\frac{\gamma_s}{a_s}f(1+p_{s-\beta}w_{s-\propto-\beta})>x_s$$

When $s\to\infty$

$$\sum_{s=e_0}^{e}\frac{\gamma_s}{a_s}f(1+p_{s-\beta}w_{s-\propto-\beta})>\infty$$

Which is a contradiction to our assumption that w_e is a non oscillatory

Hence our assumption is wrong, therefore equation (1) is oscillatory

This completes the proof.

Theorem 2 The solution of (1) is non oscillatory, if the following assumption $(H_3),(H_5)$ *and* (H_6) holds then $T_{e_0}\geq -V_n+\sum_{n=e_0}^{e}T_n^2$

Proof: Let w_e be the non oscillatory solution of (1). Without losing the generality, assume $w_e>0$.

Let $k_e=w_e-p_ew_{e-\alpha}$

$$\Delta^3(a_e\Delta k_e)+q_ef(w_{e-\beta})=0$$

$$a_{e+3}\Delta k_{e+3}-3a_{e+2}\Delta k_{e+2}+3a_{e+1}\Delta k_{e+1}-a_e\Delta k_e\leq-\rho q_e(k_{e-\beta}+p_{e-\beta}w_{e-\alpha-\beta})$$

$$\rho q_e(k_{e-\beta}+p_{e-\beta}w_{e-\alpha-\beta})>a_e\Delta k_e$$

(3) $a_e\Delta k_e-\rho q_e(k_{e-\beta}+p_{e-\beta}w_{e-\alpha-\beta})=0$

Let $T_e=\frac{k_e}{k_{e-\beta}}$

$$\Delta T_e=\Delta\left(\frac{k_e}{k_{e-\beta}}\right)=\frac{k_{e-\beta}\Delta k_e-k_e\Delta k_{e-\beta}}{k_{e+1-\beta}k_{e-\beta}}$$

$$\Delta k_e=\frac{\Delta T_ek_{e+1-\beta}k_{e-\beta}+k_e\Delta k_{e-\beta}}{k_{e-\beta}}$$

Substitute Δk_e in equation (3)

$$a_e\left(\Delta T_ek_{e+1-\beta}+\frac{k_e\Delta k_{e-\beta}}{k_{e-\beta}}\right)=\rho q_ek_{e-\beta}(1+p_{e-\beta})$$

$$a_e\Delta T_e\leq\rho q_ek_{e-\beta}(1+p_{e-\beta})-a_eT_e^2$$

$$\Delta T_e\leq\frac{1}{a_e}[\rho q_ek_{e-\beta}(1+p_{e-\beta})-a_eT_e^2]$$

Taking Summation form $n=e_0$ *to* e

$$\sum_{n=e_0}^{e}(T_{n+1}-T_n)\leq\sum_{n=e_0}^{e}\frac{1}{a_n}[\rho q_nk_{n-\beta}(1+p_{n-\beta})-a_nT_n^2]$$

$T_{e_{o+1}}-T_{e_o}+T_{e_{o+2}}-T_{e_{o+1}}-T_{e_{o+3}}-T_{e_{o+2}}-T_{e_{o+4}}-T_{e_{o+3}}+\cdots+T_{e+1}-T_e\leq\sum_{n=e_0}^{e}\frac{1}{a_n}\rho q_nk_{n-\beta}(1+p_{n-\beta})-\sum_{n=e_0}^{e}T_n^2$

$$T_{e+1}-T_{e_o}\leq V_n-\sum_{n=e_0}^{e}T_n^2$$

$$-T_{e_o}\leq V_n-\sum_{n=e_0}^{e}T_n^2$$

$$T_{e_o}\leq -V_n+\sum_{n=e_0}^{e}T_n^2$$

Hence the proof.

Theorem 3 Assume

(4) $\sum_{n=e_2}^{\infty}\left(y_eq_e-\frac{(\Delta y_e)^2a_{e+1}}{4y_e(e-e_1)^2}\right)=\infty\ for\ e_2\geq e_1$

and if the condition $(H_1)(H_7)$ *and* (H_8) hold then equation (1) is oscillatory.

Proof: Let us assume that equation (1) of $\{k_e\}$ be non oscillatory.

Without losing the generality, assume $k_e>0$ for all $e\geq e_1$

(5) $\Delta^3(a_e\Delta k_e)+q_ef(w_{e-\beta})=$

$$\Delta^3(a_e\Delta k_e)=-q_ef(w_{e-\beta})\leq 0$$

That is $\Delta^3(a_e\Delta k_e)\leq 0$ for all $e\geq e_1$

Then $\{k_e\},\{\Delta k_e\},\Delta(a_e\Delta k_e)$ and $\Delta^2(a_e\Delta k_e)$ are monotonous and actually lead to one symbol.

Claim that $\Delta^2(a_e\Delta k_e)>0$

Assume the contrary $\Delta^2(a_e\Delta k_e)\leq 0$ for all $e\geq e_2\geq e_1$

Since $\Delta^2(a_e\Delta k_e)$ is decreasing and exist a positive constant $c_1>0$ and $e_3\geq e_2$

$$\Delta^2(a_e\Delta k_e)\leq -c_1 \text{ for all } e\geq e_3$$

Taking Summation from e_3 to $(e-1)$, we get

$$\Delta\,(a_e\Delta k_e) \le \Delta\left(a_{e_3}\Delta k_{e_3}\right) - c_1(e-e_3)$$

That is $\Delta\,(a_e\Delta k_e) \to -\infty$ as $e \to \infty$

$\Delta\,(a_e\Delta k_e) \le \Delta\left(a_{e_4}\Delta k_{e_4}\right) < 0$, there exist a positive constant $c_2 > 0$ and $e_4 \ge e_3$

$$\Delta\,(a_e\Delta k_e) \le -c_2 \text{ for all } e \ge e_4$$

Taking Summation from e_4 to $(e-1)$, we get

$$\Delta\,(a_e\Delta k_e) \le \Delta\left(a_{e_4}\Delta k_{e_4}\right) - c_2(e-e_4)$$

That is $\Delta\,(a_e\Delta k_e) \to -\infty$ as $e \to \infty$

$a_e\Delta k_e \le a_{e_5}\Delta k_{e_5} < 0$, there exist a non-negative constant $c_3 > 0$ and $e_5 \ge e_4$

$$a_e\Delta k_e \le -c_3 \quad \text{for all } e \ge e_5$$

$$\Delta k_e \le -c_3\frac{1}{a_e}$$

Taking Summation from e_5 to $(e-1)$, we get

$$k_e \le k_{e_5} - c_3\sum_{n=e_5}^{e-1}\frac{1}{a_n}$$

That is $k_e \to -\infty$ as $e \to \infty$

This contradicts to the fact that k_e is positive.

Then $\Delta^2(a_e\Delta k_e) > 0$

(6) Let$t_e = \frac{y_e}{w_{e-\beta}}\Delta^2(a_e\Delta k_e) > 0$

$$\frac{t_e}{y_e} = \frac{1}{w_{e-\beta}}\Delta^2(a_e\Delta k_e)$$

$$\Delta t_e = \Delta^2(a_{e+1}\Delta k_{e+1})\Delta\left(\frac{y_e}{w_{e-\beta}}\right) + \frac{y_e}{w_{e-\beta}}\,\Delta^3(a_e\Delta k_e)$$

From (5) and (H9), we get $q_e = -\frac{\Delta^2(a_e k_e)}{w_{e-\beta}}$

$$\Delta t_e = \frac{k_{e+1}\Delta y_e - y_e k_{e+1}}{k_{e+1}k_{e+2}}\Delta^2(a_{e+1}\Delta k_{e+1}) - y_e q_e$$

$$\Delta t_e = \frac{\Delta y_e}{y_{e+1}}t_{e+1} - \frac{y_e\Delta k_{e+1}}{k_{e+2}^2}\Delta^2(a_{e+1}\Delta k_{e+1}) - y_e q_e$$

Take

$$a_e\Delta k_e = a_{e_1}\Delta k_{e_1} + \sum_{n=e_1}^{e-1}\Delta(a_e\Delta k_e) \ge (e-1-e_1)\Delta(a_e\Delta k_e)$$

$a_{e+1}\Delta k_{e+1} = a_{e_1+1}\Delta k_{e_1+1} + \sum_{n=e_1}^{e-1}\Delta(a_{s+1}\Delta k_{s+1}) \ge (e-e_1)\Delta(a_{e+1}\Delta k_{e+1})$

$\Delta(a_{e+1}\Delta k_{e+1}) = \Delta(a_{e_1+1}\Delta k_{e_1+1}) + \sum_{n=e_1}^{e-1}\Delta(a_{s+1}\Delta k_{s+1}) \ge (e-e_1)\Delta^2(a_{e+1}\Delta k_{e+1})$

$$a_{e+1}\Delta k_{e+1} \ge (e-e_1)\Delta^2(a_{e+1}\Delta k_{e+1})$$

(7) $\Delta t_e = -y_e q_e + \frac{\Delta y_e}{y_{e+1}}t_{e+1} - \frac{(e-e_1)^2 y_e}{a_{e+1}y_{e+1}^2}t_{e+1}^2$

$$\Delta t_e = -\left(y_e q_e - \frac{(\Delta y_e)^2 a_{e+1}}{4y_e(e-e_1)^2}\right)$$

Taking summation from e_2 to $e-1$, we get

$$t_e \le t_{e_1} - \sum_{n=e_2}^{e-1}\left(y_e - \frac{(\Delta y_e)^2 a_{e+1}}{4y_e(e-e_1)^2}\right)$$

Since $t_e \to -\infty$ as $e \to \infty$
Which is the contradiction $t_e > 0$
Hence the proof.

Theorem 4 Every solution of (1) is oscillatory if the conditions $(H_1)(H_7)$ and (H_8) hold, where $K_e = w_e - p_e w_{e-\propto}$

Proof: Assume the contrary that the solution of $\{w_e\}$ is non-oscillatory of (1)

w_e is equal to k_e

Without losing the generality, assume $w_e > 0$ ie) $k_e > 0$ for all $e \ge e_1$

$$\Delta^3(a_e\Delta k_e) + q_e w_{e-\beta} = 0$$

$$\Delta^3(a_e\Delta k_e) = -q_e w_{e-\beta} \le 0 \text{ for } e \ge e_1$$

Show that $\Delta^3(a_e\Delta k_e) > 0$

Suppose the contrary $\Delta^3(a_e\Delta k_e) \le 0$, $e \ge e_2 \ge e_1$

Since $\Delta^3(a_e\Delta k_e)$ is decreasing, there exist $c_1 > 0$ and $e_3 \ge e_2$

$$\Delta^3(a_e\Delta k_e) \le -c_1 \quad \text{for } e \ge e_3$$

Summing the above inequality from e_3 to $e-1$

$$\Delta^2(a_e\Delta k_e) \le \Delta^2\left(a_{e_3}\Delta k_{e_3}\right) - c_1(e-e_3)$$

That is $\Delta^2(a_e\Delta k_e) \to -\infty$ as $e \to \infty$

$\Delta^2(a_e\Delta k_e)$ is decreasing there exist $c_2 > 0$ and $e_4 \ge e_3$

$$\Delta^2(a_e\Delta k_e) \le -c_2 \quad \text{for } e \ge e_4$$

Summing the last inequality from e_4 to $e-1$

$$\Delta\,(a_e\Delta k_e) \le \Delta\left(a_{e_4}\Delta k_{e_4}\right) - c_2(e-e_4)$$

That is $\Delta\,(a_e\Delta k_e) \to -\infty$ as $e \to \infty$

$\Delta\,(a_e\Delta k_e)$ is decreasing there exist $c_3 > 0$ and $e_5 \ge e_4$

$$\Delta\,(a_e\Delta k_e) \le -c_3 \text{ for all } e \ge e_5$$

Summing the last inequality from e_5 to e – 1

$$a_e\Delta k_e \le a_{e_5}\Delta k_{e_5} - c_3(e-e_5)$$

That is $a_e\Delta k_e \to -\infty$ as $e \to \infty$

$a_e\Delta k_e$ is decreasing there exist $c_4 > 0$ and $e_6 \ge e_5$

$$a_e\Delta k_e \le -c_4 \text{ for all } e \ge e_6$$

$$\Delta k_e \le -c_4\frac{1}{a_e}$$

Summing the last inequality from e_6 to e – 1

$$\sum_{n=e_6}^{e-1}\Delta\,(a_n\Delta k_n) \le -\sum_{s=e_6}^{e-1}c_4$$

$$k_e \le k_{e_6} - c_4\sum_{s=e_6}^{e-1}\frac{1}{a_e}$$

That is $k_e \to -\infty$ as $e \to \infty$
This contradicts that k_e is positive.
Hence our assumption is wrong.
Thus the solution of equation (1) is oscillatory.

Example Consider the fourth order nonlinear neutral delay difference equation.
(8)

$$\Delta^3\left(a_e\Delta(w_e - p_e w_{e-\propto})\right) + \left(\frac{16e^3 + 96e^2 + 240e + 208}{e-3}\right)w_{e-3} = 0$$

Where $w_e = e(-1)^e, a_e = e, P_e = e^2, \alpha = 1, \beta = 3$, for all e > 3,

and $q_e = \frac{16e^3+96e^2+240e+208}{e-3}$

Equation (8) satisfies all the condition of the theorem (1). Hence all its solution are oscillatory one such solution is $e(-1)^e$.

Conclusion

In this paper, new oscillation criteria for the fourth order neutral delay difference equation are created using summation average techniques and the comparison principle. The goal of this research is to create some novel criteria for fourth order oscillatory neutral delay difference equation. So that we can use them if the other criteria don't work. We plan to expand these findings to higher order neutral delay difference equations in the future.

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