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Energy of Cartesian Product Graph Networks

Abstract. *The energy of a graph G is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of G. The Cartesian product of two graphs namely Path P_m and Double Wheel graph DW_n is constructed and its energy values on the formation of adjacency matrix, Laplacian matrix and maximum degree matrix is obtained. The upper bounds for the energy variations of different energies like graph energy, Laplacian energy and maximum degree energy of the initiated product graphs are identified and compared.*

Streszczenie. *Energia grafu G jest zdefiniowana jako suma wartości bezwzględnych wartości własnych macierzy sąsiedztwa G. Konstruowany jest iloczyn kartezjański dwóch grafów, a mianowicie Path Pm i Double Wheel graph DWn oraz jego wartości energii podczas tworzenia sąsiedztwa otrzymuje się macierz, macierz Laplace'a i macierz maksymalnego stopnia. Górne granice dla zmian energii różnych energii, takich jak energia wykresu, energia Laplace'a i maksymalny stopień energii zainicjowanych wykresów produktów, są identyfikowane i porównywane. (Energia kartezjańskich sieci grafów produktów)**()***

Keywords: Cartesian product, Eigenvalues, Graph energy, Laplacian energy, Maximum degree energy
Słowa kluczowe: Produkkt kartezjański,wartości własne, Wykres energii, Energia Laplace'a, Maksymalny stopień energii

Introduction

The electrical network analysis, dynamics and design are studied by algebraic and spectral properties of graph adjacency, Laplacian, incidence and effective resistance matrices. In history Kirchhoff's laws are most clearly formualted and analysed via graphs. A graph is an abstract representation of a set of objects called nodes or vertices in which some pairs of vertices are connected by branches or edges. The graph theoretical concepts like product of graphs and graph energies are novel areas of exploration. For basic notations and terminologies of graphs, see [6].

For any two graphs G and H, the Cartesian product $G\Box H$ [5] is defined as follows:

$$V(G\Box H) = V(G) \times V(H), \text{and}$$

$$(u_1, v_1)(u_2, v_2) \in E(G\Box H),$$

$$\Leftrightarrow u_1 = u_2 \text{ and } v_1v_2 \in E(H)$$

$$\text{or } u_1u_2 \in E(G) \text{ and } v_1 = v_2.$$

Ronan et. al [9] discussed the gracefulness of a double-wheel graph DW_m of size m consists of $2C_m + K_1$, which contains two cycles of size m, where all the vertices of two cycles are attached to a common hub. For a graph G having n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges, the adjacency matrix of $A = A(G)$ is a square matrix of order n whose $(i,j)^{th}$-entry is defined as

$$a_{ij} = \begin{cases} 1, \text{ if vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0, \text{ otherwise.} \end{cases}$$

The sum of the absolute values of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$ is defined as the energy [10] of the graph. Hence $E(G) = \sum_{i=1}^{n} |\lambda_i|$.

In [8] the faster approximation of maximum electrical flow is computed by solving linear equations in a Laplacian matrix. The Laplacian matrix [4] of $L = L(G)$ is a square matrix of order n whose $(i,j)^{th}$-entry is defined as

$$l_{ij} = \begin{cases} -1, \text{ if vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0, \text{ if vertices } v_i \text{ and } v_j \text{ are non-adjacent} \\ d_i, \text{ if i = j, where } d_i \text{ is the degree of the } i^{th} \text{ vertex.} \end{cases}$$

Then the Laplacian energy of the graph is

$$L(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|,$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the eigen values of $L(G)$.

Given d_i be the degree of the vertex $v_i, i = 1, 2, \ldots n$ of G. The maximum degree matrix of $M(G)$ is a square matrix of order n whose $(i,j)^{th}$-entry is

$$d_{ij} = \begin{cases} max\{d_i, d_j\}, \text{ if vertices } v_i \text{ and } v_j \text{ are adjacent} \\ 0, \text{ otherwise} \end{cases}$$

The maximum degree energy [10] of the graph is

$$E_M(G) = \sum_{i=1}^{n} |\mu_i|,$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the maximum degree eigen values of $M(G)$.

Ivan Gutman [3] determined the energy bounds for a simple graph G,

$$E(G) \le \frac{2m}{n} + \sqrt{(n-1)\left[2m - \left(\frac{2m}{n}\right)^2\right]}$$

while if G is k-regular,

$$E(G) \le k + \sqrt{k(n-1)(n-k)}.$$

Following this Hongzhuan [2] derived the Laplcian bounds of a simple graph G with n vertices and m edges as

$$LE(G) \le \sqrt{3mn + nm^2 - \frac{4m^2}{n}}.$$

The bounds for r-regular graph H [7] with order α where $r < \alpha - 1$ is

$$\frac{E(H)}{r + \sqrt{r(\alpha-1)(\alpha-r)}} < \epsilon, \forall \epsilon > 0.$$

Also Vladimir [11] obtained the energy bound for a non-negative matrix A with maximum entry α if $m \le n$

$$\varepsilon(A) \le \alpha\frac{(m + \sqrt{m})\sqrt{n}}{2}.$$

In this work the combination of Cartesian product of graphs and its energy estimation are considered. The nature of energy variations for various energies like graph energy, Laplacian energy and Maximum degree energy has been studied and compared. The matrix formation of graphs helps in finding the eigenvalues. It is easy to calculate the energy values if the eigenvalues of the corresponding energy matrices are known. The Cartesian product of path P_m and double wheel graph DW_n are constructed. The formulation of matrix is done from the generalized structure of these producted graphs. The eigen values are obtained from this matrix, then the energy values are calibrated and bounds are fixed. Sridhara[1] *et.al.*, has obtained the improved McClelland and Koolen-Moulton bounds for the energy of graphs. In this paper the upper bounds for energy of $P_m \Box DW_n$ are discussed.

Energy upper bounds for Cartesian product of path and double wheel graphs

Theorem: For any positive integer $m \geq 2$ and $n \geq 3$, the energy of Cartesian product of path P_m and Double wheel graph DW_n is $E(P_m \Box DW_n) \leq 4mn + min(m, n)$.

proof. The path P_m consists of m vertices and $m-1$ edges and its vertices and edges are represented by

$$V(P_m) = \{u_i : 1 \leq i \leq m\} \text{ and}$$

$$E(P_m) = \{u_i u_{i+1} : 1 \leq i \leq m-1\} \text{ respectively.}$$

The Double wheel graph DW_n consists of $2n+1$ vertices and $4n$ edges and its vertices and edges are defined as

$$V(DW_n) = \{v_0\} \cup \{v_i : 1 \leq i \leq n\} \cup \{v_k : n+1 \leq k \leq 2n\}$$

$$\begin{aligned} E(DW_n) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\} \cup \{e'_k\} \\ \cup \{e''_i : n+1 \leq i \leq 2n\} \\ \cup \{e'''_i : n+1 \leq i \leq 2n\} \cup \{e'''_k\} \end{aligned}$$

where edges e_i connect the vertices $v_0 v_i$ $(1 \leq i \leq n)$, e'_i are the edges between $v_i v_{i+1} (1 \leq i \leq n-1)$, e'_k is the edge between v_n and v_1, e''_i are the edges between vertices $v_0 v_i (n+1 \leq i \leq 2n)$, e'''_i are the edges between the vertices $v_i v_{i+1} (n+1 \leq i \leq 2n-1)$ and e'''_k is the edge between v_{2n} and v_{n+1}.

By applying the Cartesian product for these two graphs, it establishes a new product graph with $m(2n+1)$ vertices and $4mn + min(m, n)$ edges such that

$$V(P_m \Box Dw_n) = \{u_i v_j : 1 \leq i \leq m, 0 \leq j \leq 2n\},$$

$$\begin{aligned} E(P_m \Box Dw_n) = \{(u_i v_0, u_i v_j) : 1 \leq i \leq m, \\ 1 \leq j \leq 2n\} \\ \cup \{(u_i v_j, u_i v_{j+1}) : 1 \leq i \leq m, \\ 1 \leq j \leq n-1\} \\ \cup \{(u_i v_n, u_i v_1) : 1 \leq i \leq m\} \\ \cup \{(u_i v_j, u_i v_{j+1}) : 1 \leq i \leq m, \\ n+1 \leq j \leq 2n-1\} \\ \cup \{(u_i v_{2n}, u_i v_{n+1}) : 1 \leq i \leq m\} \\ \cup \{(u_i v_j, u_{i+1} v_j) : 1 \leq i \leq m-1, \\ 0 \leq j \leq 2n\} \end{aligned}$$

The adjacency matrix of $P_m \Box Dw_n$ is given by

$$A(P_m \Box Dw_n) = \begin{cases} 1, \ if \ i \text{ and } j \text{ are adjacent} \\ 0, \ if \ i \text{ and } j \text{ are non - adjacent} \end{cases}$$

and it is structured as a block matrix as follows

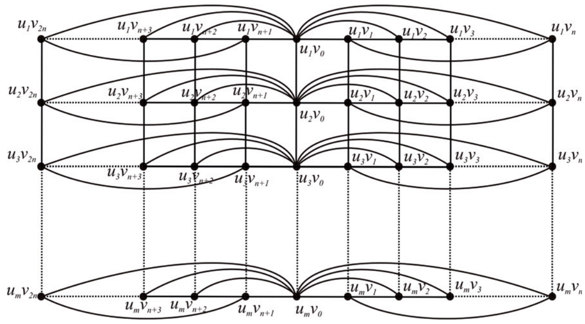

Fig. 1. Generalized Cartesian product of $P_m \Box DW_n$

$$A(P_m \Box Dw_n) = \left[\begin{array}{c|c|c|c|c|c} S & I & O & \dots & O & O \\ \hline I & S & I & \dots & O & O \\ \hline O & I & S & \dots & O & O \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline O & O & O & \dots & S & I \\ \hline O & O & O & \dots & I & S \end{array}\right]$$

The formulation of the block matrix $S = [s_{ij}]_{2n+1 \times 2n+1}$ is

$$s_{ij} = \begin{cases} 1, \text{if } i = 0, 1 \leq j \leq 2n; \\ \quad j = 0, 1 \leq i \leq 2n; \\ \quad i = 1, j = 2 \text{ and } j = n; \\ \quad i = n+1, j = n+2 \text{ and } j = 2n; \\ \quad j = 1, i = 2 \text{ and } i = n; \\ \quad j = n+1, i = n+2 \text{ and } i = 2n; \\ \quad 2 \leq i \leq n-1, j = i+1; \\ \quad n+2 \leq i \leq 2n-1, j = i+1; \\ \quad 2 \leq j \leq n-1, i = j+1; \\ \quad n+2 \leq j \leq 2n-1, i = j+1; \\ 0, \text{otherwise}, \end{cases}$$

Therefore the adjacency matrix of S is constructed equally

$$\begin{array}{c} \\ v_0 \\ v_1 \\ \vdots \\ v_n \\ v_{n+1} \\ v_{n+2} \\ \vdots \\ v_{2n} \end{array} \begin{array}{c} \begin{array}{cccccccc} v_0 & v_1 & \cdots & v_n & v_{n+1} & v_{n+2} & \cdots & v_{2n} \end{array} \\ \begin{pmatrix} 0 & 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 & 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \end{pmatrix} \end{array}$$

The blocks I is the identity matrix of order $2n+1$ and O is the zero matrix of order $2n+1$.

Set $det\,(A(P_m \Box Dw_n) - \lambda I) = 0$. The characteristic equation of this adjacency matrix with order $m(2n+1)$ is of the form

$$(-\lambda)^{m(2n+1)} + tr(-\lambda)^{m(2n+1)-1} + \ldots + det(A) = 0$$

which has exactly $m(2n+1)$ roots and let it be

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{m(2n+1)}.$$

The energy $E = \sum_{i=1}^{m(2n+1)} |\lambda_i|$

Applying Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{m(2n+1)} |\lambda_i|\right)^2 \leq \sum_{i=1}^{m(2n+1)} |1| \sum_{i=1}^{m(2n+1)} |\lambda_i|^2$$

$$\left(\sum_{i=2}^{m(2n+1)-1} |\lambda_i| - |\lambda_1| - |\lambda_{m(2n+1)}|\right)^2$$

$$\leq \left(\sum_{i=2}^{m(2n+1)-1} |1| - 2\right)$$

$$\cdot \left(\sum_{i=2}^{m(2n+1)-1} |\lambda_i|^2 - |\lambda_1|^2 - |\lambda_{m(2n+1)}|^2\right)$$

$$\sum_{i=2}^{m(2n+1)-1} |\lambda_i| \leq |\lambda_1| + |\lambda_{m(2n+1)}|$$

$$+ \sqrt{\{m(2n+1) - 3\}}$$

$$\cdot \sqrt{\left(\sum_{i=2}^{m(2n+1)-1} |\lambda_i|^2 - |\lambda_1|^2 - |\lambda_{m(2n+1)}|^2\right)}$$

Therefore

(1) $$E(G) \leq |\lambda_1| + |\lambda_{m(2n+1)}|$$

$$+ \sqrt{\{m(2n+1) - 3\}}$$

$$\cdot \sqrt{\left(\sum_{i=2}^{m(2n+1)-1} |\lambda_i|^2 - |\lambda_1|^2 - |\lambda_{m(2n+1)}|^2\right)}$$

Substituting $|\lambda_1| = x$, $|\lambda_{m(2n+1)}| = y$ and multiplying both sides of (1) with $\frac{1}{\sqrt{m(2n+1)-1}}$

$$\frac{1}{\sqrt{m(2n+1)-1}} E(G) \leq \frac{1}{\sqrt{m(2n+1)-1}}$$

$$\cdot [x + y + \sqrt{\{m(2n+1) - 3\}}$$

$$\cdot \sqrt{\left(\sum_{i=2}^{m(2n+1)-1} |\lambda_i|^2 - x^2 - y^2\right)}]$$

Hence the function is

(2) $$f(x, y) = \frac{1}{\sqrt{m(2n+1)-1}} \cdot [x + y + \sqrt{\{m(2n+1) - 3\}} \cdot \sqrt{\{4mn + min(m, n)\}^2 - x^2 - y^2}]$$

Differentiating (2) partially with respect to x upto the second order derivative

$$f_{xx} = -\frac{\sqrt{m(2n+1)-3}\left[\{4mn+min(m,n)\}^2-y^2\right]}{\sqrt{m(2n+1)-1}[\{4mn+min(m,n)\}^2-x^2-y^2]^{\frac{3}{2}}}$$

$$f_{yy} = -\frac{\sqrt{m(2n+1)-3}\left[\{4mn+min(m,n)\}^2-x^2\right]}{\sqrt{m(2n+1)-1}[\{4mn+min(m,n)\}^2-x^2-y^2]^{\frac{3}{2}}}$$ and

$$f_{xy} = -\frac{\sqrt{m(2n+1)-3}\ xy}{\sqrt{m(2n+1)-1}[\{4mn+min(m,n)\}^2-x^2-y^2]^{\frac{3}{2}}}$$

The stationary points of the function are

$$x = y = \frac{1}{\sqrt{m(2n+1)-1}} \{4mn + min(m, n)\}.$$

The value of second order derivatives at these points are

$$f_{xx} = f_{yy} = -\frac{m(2n+1)-1}{[m(2n+1)-3][4mn+min(m,n)]} \leq 0$$

$$f_{xy} = -\frac{1}{[m(2n+1) - 3]\,[4mn + min(m, n)]} \leq 0$$

$$\Delta = \frac{m(2n+1)}{[m(2n+1) - 3]^2\,[4mn + min(m, n)]^2} \geq 0$$

The maximum value of the function is

$$f\left(\frac{4mn + min(m, n)}{\sqrt{m(2n+1)-1}}, \frac{4mn + min(m, n)}{\sqrt{m(2n+1)-1}}\right)$$

$$= \frac{1}{\sqrt{m(2n+1)-1}} \cdot [\frac{2\{4mn + min(m, n)\}}{\sqrt{m(2n+1)-1}} + \frac{\{m(2n+1) - 3\}\{4mn + min(m, n)\}}{\sqrt{m(2n+1)-1}}]$$

$$= 4mn + min(m, n)$$

Thus from (1)

$$f\left(\frac{4mn + min(m, n)}{\sqrt{m(2n+1)-1}}, \frac{4mn + min(m, n)}{\sqrt{m(2n+1)-1}}\right) \leq 4mn + min(m, n).$$

Hence $E(P_m \Box DW_n) \leq 4mn + min(m, n)$.

Illustrations: The following table illustrates the energy upper bounds for the Cartesian product of P_m and DW_n.

Graphs $P_m \Box DW_n$	Vertices m(2n+1)	Edges 4mn+(m-1)(2n+1)	Energy	Upper bound
$P_2 \Box DW_4$	18	41	32	34
$P_3 \Box DW_3$	21	50	37.1882	39
$P_4 \Box DW_5$	44	113	78.3101	84
$P_5 \Box DW_3$	35	88	62.4865	63
$P_6 \Box DW_4$	54	141	99.9517	100
$P_7 \Box DW_6$	91	246	168.6693	174
$P_8 \Box DW_9$	120	329	222.3449	231
$P_9 \Box DW_8$	153	424	283.6168	296
$P_{10} \Box DW_{10}$	210	589	388.7023	410
$P_{25} \Box DW_{25}$	1275	3724	2314.8	2525

Table 1. Energy and energy upper bounds for $P_m \Box DW_n$.

Theorem: For any positive integer $m \geq 2$ and $n \geq 3$, the Laplacian energy of Cartesian product of path P_m and double wheel graph DW_n is

$$LE(P_m \Box DW_n) \leq m(2n+1)(m + n + 4).$$

proof. The Laplacian matrix of $P_m \square Dw_n$ is given by

$$L(P_m \square Dw_n) = \begin{cases} 2n+1, \text{ if } i=0, j=0 \\ \quad \text{and } i=n, j=n \\ 2n+2, \text{ if } i=j, 1 \le i \le 2n-1, \\ \quad 1 \le j \le 2n-1 \\ -1, \text{ if } i \text{ and } j \text{ are adjacent} \\ 0, \text{ if } i \text{ and } j \text{ are non - adjacent} \end{cases}$$

and it is structured as a block matrix as follows

$$L(P_m \square Dw_n) = \left[\begin{array}{c|c|c|c|c|c} R_1 & D & O & \dots & O & O \\ \hline D & R & D & \dots & O & O \\ \hline O & D & R & \dots & O & O \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline O & O & O & \dots & R & D \\ \hline O & O & O & \dots & D & R_2 \end{array}\right]$$

Here the block matrix $R = [r_{ij}]_{2n+1 \times 2n+1}$ is formulated by

$$r_{ij} = \begin{cases} 2n+2, \text{ if } i=j, 0 \le i \le 2n, \\ \quad 0 \le j \le 2n \\ -1, \text{if } i=1, 1 \le j \le 2n; \\ \quad j=1, 1 \le i \le 2n; \\ \quad i=1, j=2 \text{ and } j=n; \\ \quad i=n+1, j=n+2 \text{ and } j=2n; \\ \quad j=1, i=2 \text{ and } i=n; \\ \quad j=n+1, i=n+2 \text{ and } i=2n; \\ \quad 2 \le i \le n-1, j=i+1; \\ \quad n+2 \le i \le 2n-1, j=i+1; \\ \quad 2 \le j \le n-1, i=j+1; \\ \quad n+2 \le j \le 2n-1, i=j+1; \\ 0, \text{otherwise}, \end{cases}$$

Therfore the structure of block R in the Laplacian matrix equals

$$\begin{array}{c} \\ v_0 \\ v_1 \\ \vdots \\ v_n \\ \vdots \\ v_{2n} \end{array} \begin{array}{c} \begin{array}{cccccc} v_0 & v_1 & \cdots & v_n & \cdots & v_{2n} \end{array} \\ \begin{pmatrix} 2n+2 & -1 & \dots & -1 & \dots & -1 \\ -1 & 2n+2 & \dots & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & 2n+2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 0 & \dots & 2n+2 \end{pmatrix} \end{array}$$

The block matrix $R_1 = [r_{(1)ij}]_{2n+1 \times 2n+1}$ resembles R where

$$r_{(1)ij} = \begin{cases} 2n+1, \text{ if } i=0, j=0 \\ \text{All the remaining adjacency positions} \\ \text{are similar as in block matrix } R \end{cases}$$

The structure of block matrix R_1 is

$$\begin{array}{c} \\ v_0 \\ v_1 \\ \vdots \\ v_n \\ \vdots \\ v_{2n} \end{array} \begin{array}{c} \begin{array}{cccccc} v_0 & v_1 & \cdots & v_n & \cdots & v_{2n} \end{array} \\ \begin{pmatrix} 2n+1 & -1 & \dots & -1 & \dots & -1 \\ -1 & 2n+2 & \dots & -1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & 2n+2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 0 & \dots & 2n+2 \end{pmatrix} \end{array}$$

Similarly for the block matrix $R_2 = [r_{(2)ij}]_{2n+1 \times 2n+1}$ is

$$r_{(2)ij} = \begin{cases} 2n+1, \text{ if } i=2n, j=2n \\ \text{The remaining adjacency positions} \\ \text{are same as mentioned in block matrix } R \end{cases}$$

Therefore the adjacency block matrix of R_2 equals

$$\begin{array}{c} \\ v_0 \\ v_1 \\ . \\ v_n \\ \vdots \\ v_{2n} \end{array} \begin{array}{c} \begin{array}{cccccc} v_0 & v_1 & \cdots & v_n & \cdots & v_{2n} \end{array} \\ \begin{pmatrix} 2n+2 & -1 & \dots & -1 & \dots & -1 \\ -1 & 2n+2 & \dots & -1 & \dots & 0 \\ & & & & & \vdots \\ -1 & -1 & \dots & 2n+2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ -1 & 0 & \dots & 0 & \dots & 2n+1 \end{pmatrix} \end{array}$$

The blocks $D = [d_{ij}]_{2n+1 \times 2n+1}$ is the diagonal matrix where

$$d_{ij} = \begin{cases} -1, \text{if } i=j \\ 0, \text{otherwise}, \end{cases}$$

and the blocks O are the null matrix of order $2n+1$.

The method of proving the upper bound for this Laplacian energy of $P_m \square DW_n$ is similar to the ordinary energy discussed.

Illustrations: The following table illustrates the Laplcian energy upper bounds for the Cartesian product of P_m and DW_n.

Graphs $P_m \square DW_n$	Vertices m(2n+1)	Edges 4mn+(m-1)(2n+1)	Laplacian Energy	Upper bound
$P_2 \square DW_4$	18	41	178	180
$P_3 \square DW_3$	21	50	166	210
$P_4 \square DW_5$	44	113	526	572
$P_5 \square DW_3$	35	88	278	420
$P_6 \square DW_4$	54	141	538	756
$P_7 \square DW_6$	91	246	1272	1547
$P_8 \square DW_9$	120	329	1918	2280
$P_9 \square DW_8$	153	424	2752	3213
$P_{10} \square DW_{10}$	210	589	4618	5040
$P_{25} \square DW_{25}$	1275	3724	66298	68850

Table 2. Laplacian energy and its upper bounds for $P_m \square DW_n$.

Theorem: For any positive integer $m \ge 2$ and $n \ge 3$, the Maximum degree energy is

$$E_M(P_m \square DW_n) \le m(2n+1)(m+n+3)+26.$$

proof. The Maximum degree matrix of $P_m \square Dw_n$ is given by

$$M(P_m \square Dw_n) = \begin{cases} max\{4, 5, \kappa, \ell\}, \\ \quad \text{if } i \text{ and } j \text{ are adjacent} \\ \text{where } \kappa = 2n+1 \text{ and } \ell = 2n+2 \\ 0, \text{ if } i \text{ and } j \text{ are non - adjacent} \end{cases}$$

and formulated as a block matrix as follows

$$M(P_m\Box Dw_n) = \left[\begin{array}{c|c|c|c|c|c} K & D & O & \dots & O & O \\ \hline D & L & D & \dots & O & O \\ \hline O & D & L & \dots & O & O \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline O & O & O & \dots & L & D \\ \hline O & O & O & \dots & D & K \end{array}\right]$$

Here the block matrix $K = [k_{ij}]_{2n+1\times 2n+1}$ where

$$k_{ij} = \begin{cases} \kappa, \text{ if } i = 0, 1 \leq j \leq 2n \\ \quad\quad \text{and } j = 0, 1 \leq j \leq 2n, \\ 4, \text{ if } i = 1, j = 2 \text{ and } j = n; \\ \quad i = n+1, j = n+2 \text{ and } j = 2n; \\ \quad j = 1, i = 2 \text{ and } i = n; \\ \quad j = n+1, i = n+2 \text{ and } i = 2n; \\ \quad 2 \leq i \leq n-1, j = i+1; \\ \quad n+2 \leq i \leq 2n-1, j = i+1; \\ \quad 2 \leq j \leq n-1, i = j+1; \\ \quad n+2 \leq j \leq 2n-1, i = j+1; \\ 0, \text{otherwise}, \end{cases}$$

The block matrix K in the Maximum degree matrix equals

$$\begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_n \\ v_{n+1} \\ v_{n+2} \\ \vdots \\ v_{2n-1} \\ v_{2n} \end{array} \begin{array}{c} \begin{array}{ccccccccc} v_0 & v_1 & v_2 & \cdots & v_n & v_{n+1} & v_{n+2} & \cdots & v_{2n} \end{array} \\ \left(\begin{array}{ccccccccc} 0 & \kappa & \kappa & \dots & \kappa & \kappa & \kappa & .... & \kappa \\ \kappa & 0 & 4 & \dots & 4 & 0 & 0 & \dots & 0 \\ \kappa & 4 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \kappa & 4 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \kappa & 0 & 0 & \dots & 0 & 0 & 4 & \dots & 4 \\ \kappa & 0 & 0 & \dots & 0 & 4 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \kappa & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 4 \\ \kappa & 0 & 0 & \dots & 0 & 4 & 0 & \dots & 0 \end{array}\right) \end{array}$$

Also the block matrix $L = [l_{ij}]_{2n+1\times 2n+1}$ where

$$l_{ij} = \begin{cases} \ell, \text{ if } i = 0, 1 \leq j \leq 2n \\ \quad\quad \text{and } j = 0, 1 \leq j \leq 2n, \\ 5, \text{ if } i = 1, j = 2 \text{ and } j = n; \\ \quad i = n+1, j = n+2 \text{ and } j = 2n; \\ \quad j = 1, i = 2 \text{ and } i = n; \\ \quad j = n+1, i = n+2 \text{ and } i = 2n; \\ \quad 2 \leq i \leq n-1, j = i+1; \\ \quad n+2 \leq i \leq 2n-1, j = i+1; \\ \quad 2 \leq j \leq n-1, i = j+1; \\ \quad n+2 \leq j \leq 2n-1, i = j+1; \\ 0, \text{otherwise}, \end{cases}$$

Hence the block matrix L in the Maximum degree matrix is constructed as

$$\begin{array}{c} \\ v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_n \\ v_{n+1} \\ v_{n+2} \\ \vdots \\ v_{2n-1} \\ v_{2n} \end{array} \begin{array}{c} \begin{array}{ccccccccc} v_0 & v_1 & v_2 & \cdots & v_n & v_{n+1} & v_{n+2} & \cdots & v_{2n} \end{array} \\ \left(\begin{array}{ccccccccc} 0 & \ell & \ell & \dots & \ell & \ell & \ell & \dots & \ell \\ \ell & 0 & 5 & \dots & 5 & 0 & 0 & \dots & 0 \\ \ell & 5 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell & 5 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \ell & 0 & 0 & \dots & 0 & 0 & 5 & \dots & 5 \\ \ell & 0 & 0 & \dots & 0 & 5 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \ell & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 5 \\ \ell & 0 & 0 & \dots & 0 & 5 & 0 & \dots & 0 \end{array}\right) \end{array}$$

The blocks $D = [d_{ij}]_{2n+1\times 2n+1}$ is the diagonal matrix where

$$d_{ij} = \begin{cases} 5, \text{if } i = j \\ 0, \text{otherwise}, \end{cases}$$

and the blocks O are the null matrix of order $2n+1$.

The proof of Maximum degree energy for $P_m\Box DW_n$ is also similar to Theorem.1 and its bounds between varoius paths and double wheels is calibrated in the illustration.

Illustrations: The following table illustrates the Maximum degree energy upper bounds for the Cartesian product of P_m and DW_n.

Graphs $P_m\Box DW_n$	Vertices m(2n+1)	Edges 4mn+(m-1)(2n+1)	Maximum Degree Energy	Upper bound
$P_2\Box DW_4$	18	41	183.0728	188
$P_3\Box DW_3$	21	50	209.5852	215
$P_4\Box DW_5$	44	113	539.1855	554
$P_5\Box DW_3$	35	88	364.7040	411
$P_6\Box DW_4$	54	141	640.6140	728
$P_7\Box DW_6$	91	246	1237	1482
$P_8\Box DW_9$	120	329	1721.7	2186
$P_9\Box DW_8$	153	424	2300	3086
$P_{10}\Box DW_{10}$	210	589	3400.8	4856
$P_{25}\Box DW_{25}$	1275	3724	28048	65575

Table 3. Maximum degree energy and its upper bounds of $P_m\Box DW_n$.

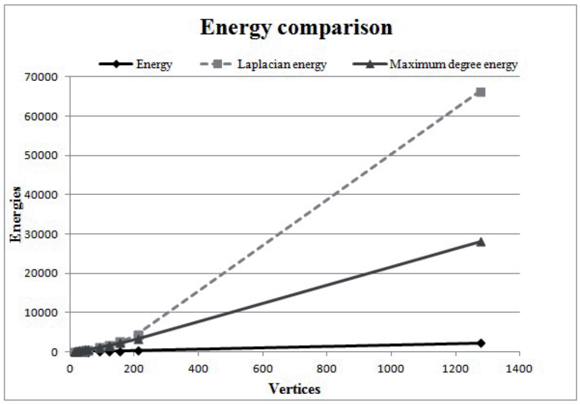

Fig. 2. Graphical representation of variation in energies

Conclusion

This research proposes a methodical approach to capture the essence of the graph theoretical concepts like product of graphs, graph energy and to evaluate the various energies and estimate their bounds. It is hard to fix the bounds of these graphs due to mass and complexity of data. Many researchers obtained the improved bounds of different graphs. In this work three different types of energies are studied and compared for cartesian product of path and double wheel graphs. Among the three graph energies examined, the Laplacian energy leads the other two energies as shown in Figure 2. Furthermore bounds can be achieved for other energies of different products between various graphs.

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REFERENCES

[1] G. Sridhara, M. R. Rajesh Kanna, R. Jagadeesh, I. N. Cangul.: Improved McClelland and Koolen-Moulton bounds for the energy of graphs, Scientia Magna, 13(1), pp. 1–10, 2018.
[2] Hongzhuan Wang, Hongbo Hua.: Note on Laplacian energy of graphs, MATCH Communications in Mathematical and in Computer Chemistry, 59, pp. 373–380, 2008.
[3] I. Gutman.: The energy of a graph, Ber. Math-Statist. Sekt. Forschungszentrum Graz, 103, pp. 1–22, 1978.
[4] Kinkar Ch. Das, Ivan Gutman, A. Sinan Çevik Bo Zhou.: On Laplacian energy, MATCH Communications in Mathematical and in Computer Chemistry, 70, pp. 689–696, 2013.
[5] Marcin Pilipczuk, Michal Pilipczuk, Riste Škrekovski.: Some results on Vizing's conjecture and related problems, Discrete Applied Mathematics, 160, pp. 2484–2490, 2012.
[6] R. Balakrishnan, K. Ranganathan.: A Textbook of Graph Theory, Springer, New York, 2000.
[7] R. Balakrishnan.: The energy of a graph, Linear Algebra and its Applications, 387, pp. 287–295, 2004.
[8] Paul Christiano, Jonathan A. Kelner, Aleksander Madry, Daniel Spielman.: Electrical Flows, Laplacian Systems, and Faster Approximation of Maximum Flow in Undirected Graphs, Proceedings of the forty-third annual ACM symposium on Theory of Computing ,pp. 273–282, 2011.
[9] Ronan Le Bras, Carla P. Gomes, Bart Selman.: Double-Wheel Graphs are Graceful, Proceedings of the Twenty third International Joint Conference on Artificial Intelligence, pp. 587–593, 2013.
[10] S.Meenakshi, S. Lavanya.: A Survey on Energy of Graphs, Annals of Pure and Applied Mathematics,8(2),pp. 183–191, 2014.
[11] Vladimir Nikiforov.: The energy of graphs and matrices, Journal of Mathematical Analysis and Applications,326, pp. 1472–1475, 2007.