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Optimization of the polynomial fifth-order interpolation 1P kernel in the time domain

***Abstract**. In the first part of this paper, the fifth-order polynomial interpolation convolution one-parameter kernel, is presented. After that, optimization of the interpolation kernel in the time domain was performed. The optimization criterion was the minimization of the interpolation error. The minimization of the error was realized by choosing the optimal value of the kernel parameter α_{opt}. Verification of the correctness of the selection of the α_{opt}, by experiment was performed. First, test functions with a complex time shape were created. After that, the test functions were interpolated using interpolation kernels with some analysed kernel parameters. Interpolation errors are shown using MSE. Finally, by applying a comparative analysis, the verification of the choice of the optimal kernel parameter α_{opt} was carried out.*

***Streszczenie.** W pierwszej części artykułu zaprezentowano jednoparametrowe jądro splotu interpolacji wielomianowej piątego stopnia. Następnie przeprowadzono optymalizację jądra interpolacyjnego w dziedzinie czasu. Kryterium optymalizacji stanowiła minimalizacja błędu interpolacji. Minimalizację błędu realizowano poprzez dobór optymalnej wartości parametru jądra α_{opt}. Weryfikację poprawności wyboru α_{opt} przeprowadzono metodą eksperymentalną. W pierwszej kolejności utworzono funkcje testowe o złożonym kształcie czasu. Następnie funkcje testowe interpolowano za pomocą jąder interpolacyjnych z niektórymi analizowanymi parametrami jądra. Błędy interpolacji są pokazywane za pomocą MSE. Na koniec, stosując analizę porównawczą, przeprowadzono weryfikację wyboru optymalnego parametru jądra α_{opt}. (**Optymalizacja jądra wielomianowej interpolacji piątego rzędu 1P w dziedzinie czasu**)*

Keywords: Convolution, Interpolation, Polynomial kernel, Taylor series.
Słowa kluczowe: Splot, interpolacja, jądro wielomianu, szereg Taylora.

Introduction

In digital signal processing the application of interpolation is very current [1]. Spatial transformations (rotations, translations, changes image dimensions, geometric deformations, etc.) are often performed in digital image processing [2, 3]. With these transformations, it is necessary to determine the intensity of pixels whose spatial coordinates are outside of the grid [4, 5]. These problems are solved by applying interpolations in the spatial domain [6]. When processing audio signals in the time domain (resampling), interpolation is performed in the time domain [7]. Audio processing, and especially speech signal processing (fundamental frequency estimation), requires interpolation in the spectral domain.

Most often, interpolation should be realized in real-time. The application of numerical interpolation formulas (Lagrangian, Newtonian, Gaussian, Stirling, Bessel, Chebyshev,...) requires knowledge of a large amount of data, sometimes the complete signal. For this reason, interpolation formulas are often of an impractically large order, that is, of great numerical complexity. The consequence of the high numerical complexity is an impractically long interpolation time.

In order to increase the speed of interpolation, which is a fundamental requirement for application in real-time systems, convolutional interpolation is intensively applied. Convolutional interpolation is realized by convolution between the discrete signal and the continuous interpolation kernel. The precision and speed of interpolation are directly dependent on the interpolation kernel. Theoretical analysis showed that the interpolation kernel, with the time-spatial form $r = sin(x)/x$, is the ideal interpolation kernel for interpolating band-limited discrete signals [8]. This kernel is referred in the scientific literature as $sinc$. The spectral characteristic of the $sinc$ kernel is in the form of a box function. The properties of the box spectral characteristic are: a) in the pass-band is flat and equal to one, b) in the stop-band is flat and equal to zero, and c) with an ideal slope in the transition area [9]. Interpolation $sinc$ kernel is defined in the range $(-\infty \leq x \leq +\infty)$. This fact indicates that it is not possible to practically realize the $sinc$ kernel [10].

In order to enable the practical realization of the $sinc$ kernel, kernel can be truncated to the length L using a window. The truncation process is called windovization. However, simple kernel windovization leads to negative consequences (the spectral characteristic has a ripple in the pass-band and stop-band, as well as a finite slope in the transition range) which leads to a decrease in the precision of the interpolation. In the last thirty years, the current task at DSP is the construction of an interpolation kernel using a function of low numerical complexity, which will satisfactorily approximate $sinc$ on the interval L. The main direction is towards the construction of low-order $(n \leq 7)$ polynomial kernels [11]. A large number of polynomial kernels have been proposed in the scientific literature. Numerically the simplest is the polynomial zeroth-order kernel. Interpolation is performed by rounding to the nearest-neighbor sample [12]. In addition to high execution speed, interpolation with this kernel leads to the appearance of a large interpolation error e. A linear, polynomial first-order interpolation kernel is described in [13]. A cubic, polynomial third-order interpolation kernel, is described in [8]. Convolutional interpolation using the third-order kernel is more precise than the previous two kernels.

The parameterization of the polynomial third-order kernel was proposed by Robert Keys in [8]. By inserting the parameter α into the coefficients of the kernel, the parameterization is performed. A very significant fact is that, by changing the kernel parameter α, the kernel can be adapted to the specific signal, and, in this way, the interpolation precision can be increased. Later, in the scientific literature, this kernel, in honor of the author who proposed it, the one-parameter Keys (1P Keys) kernel was named. In order to further increase the precision of interpolation, kernels of the third-order with two parameters (2P Keys, parameters α and β) were proposed [14]. A further increase in interpolation precision was achieved by constructing a three-parameter (α, β, γ) 3P Keys kernel [15]. A fifth-order polynomial one-parameter interpolation kernel is described in [16]. The length of the proposed kernel is $L = 6$. Optimization of the kernel parameter α in the spectral domain was performed. The optimization criterion was the reduction of ripple of the spectral characteristic in pass-band and stop-band. Through the optimization process, the optimal kernel parameter, α = 3/64, was determined.

This paper presents the results of the optimization of the polynomial fifth-order 1P interpolation kernel. The optimization was performed in the time domain. The optimization criterion is the minimization of the interpolation error e. First, the interpolation function g is determined. After that, assuming that the function f, which is to be interpolated, has at least five continuous derivatives in the interval where the interpolation is performed, the development of the function f in Taylor series is performed. The Taylor series has been expanded to the fifth term. Then the interpolation error $e = f - g$ was formed. Finally, the minimization of the interpolation error was realized, so that the f and interpolated function g agree up to the fifth term in the Taylor series expansion. The minimization was achieved by choosing the optimal value of the kernel parameter among seven kernel parameter values, which, potentially, could represent the optimal choice (α_{opt}).

With the aim of verifying the correctness of the choice of the optimal kernel parameter α_{opt}, an experiment was carried out. First, three functions (f_1, f_2 and f_3), which represent signals of complex time form, are created. After that, the functions are interpolated using the fifth-order interpolation 1P kernel. In the 1P kernel the analyzed kernel parameters are implemented. After interpolation the interpolation errors e were calculated. Based on them, the mean squared errors MSE are determined. Comparative analysis of MSE showed that α_{opt} was correctly determined, and, therefore, the verification was performed. The results of the experiment are presented using graphs and tables.

Fifth-order polynomial interpolation 1P kernel.

In the paper [16] convolutional, one-parameter fifth-order polynomial interpolation kernel, is described. The 1P kernel is defined on the interval (-3, 3) and approximates the ideal *sinc* interpolation kernel. Outside of the interval (-3, 3) the interpolation kernel is zero. The 1P kernel is composed of piecewise fifth-order polynomials, which are defined on the subintervals (-3, -2), (-2, -1), (-1,0), (0,1), (1, 2), and (2,3). Therefore, the length of the kernel is $L = 6$. The kernel r is defined by:

$$r = \begin{cases} (10\alpha - 21/16)|s|^5 - (18\alpha - 45/16)|s|^4 & \\ +(8\alpha - 5/2)|s|^2 + 1, & 0 \le |s| < 1, \\ (11\alpha - 5/16)|s|^5 - (88\alpha - 45/16)|s|^4 & \\ +(270\alpha - 10)|s|^3 - (392\alpha - 35/2)|s|^2 & 1 \le |s| < 2, \\ +(256\alpha - 15)|s| - 66\alpha + 5, & \\ \alpha|s|^5 - 14\alpha|s|^4 + 78\alpha|s|^3 - 216\alpha|s|^2 & 2 \le |s| < 3, \\ +297\alpha|s| - 162\alpha, & \\ 0, & 3 \le |s|. \end{cases} \quad (1)$$

where α is the kernel parameter.

The kernel parameter α directly affects the time-spectral characteristics of the 1P kernel. Changing the value of the kernel parameter α affects the interpolation precision. By minimizing the interpolation error e, it is possible to determine the optimal value of the kernel parameter, α_{opt}, and, in this way, optimize the interpolation kernel r. In the paper [16], the optimization of the 1P kernel in the spectral domain was performed. As an optimization criterion, the condition of eliminating the ripple of the spectral characteristic H was set.

In the rest of this paper, the optimization of the 1P kernel, which was performed in the time domain, is presented. The optimization criterion was the minimization of the interpolation error e.

Optimization of the interpolation 1P kernel

The interpolation function $g(x)$ is a special type of approximation function. Its fundamental property is that it is equal to the sampled data, that is, the values of the function $f(x)$ in the interpolation nodes. Then $g(x_k) = f(x_k)$, where $0 \le k \le N - 1$, and N is the total number of interpolation nodes, in the segment where the function is interpolated. Let us assume that x is a point, in which the interpolation of the function $f(x)$ should be performed. Let x be between two consecutive interpolation nodes, denoted as x_j and x_{j+1}. Let $s = (x - x_j)/h$. Then $(x - x_k)/h = (x - x_j + x_j - x_k)/h = s + j + k$. The interpolation, that is, the reconstructed function $g(x)$, is determined by convolutional interpolation [8] of the interpolation function $f(x)$ with the interpolation kernel r:

$$g(x) = \sum_k c_k r\left(\frac{x - x_k}{h}\right) = \sum_k c_k r(s + j - k), \quad (2)$$

where c_k is the value of the function $f(x)$ in the interpolation k-th node (k-th sample), and h is the sampling increment. By developing the sum from (2), the reconstruction function can be written as:

$$\begin{aligned} g(x) &= c_{j-2} r(s+2) + c_{j-1} r(s+1) + c_j r(s) \\ &+ c_{j+1} r(s-1) + c_{j+2} r(s-2) + c_{j+3} r(s-3) \end{aligned}. \quad (3)$$

The value of kernel r, for the segment is $-3 \le s < -2$, is:

$$r(s+2) = \alpha s^5 - 4\alpha s^4 + 6\alpha s^3 - 4\alpha s^2 + \alpha s. \quad (4)$$

Continuing this procedure, the kernel values in the other segments are determined:

$$\begin{aligned} r(s+1) &= (11\alpha - 5/16)s^5 + (5/4 - 33\alpha)s^4 \\ &+ (28\alpha - 15/8)s^3 + 5/4 \cdot s^2 + (-\alpha - 5/16)s \end{aligned}, \quad (5)$$

$$\begin{aligned} r(s) &= (10\alpha - 21/16)s^5 \\ &+ (45/16 - 18\alpha)s^4 + (8\alpha - 5/2)s^2 + 1 \end{aligned}, \quad (6)$$

$$\begin{aligned} r(s-1) &= (21/16 - 10\alpha)s^5 + (32\alpha - 15/4)s^4 \\ &+ (15/8 - 28\alpha)s^3 + 5/4 \cdot s^2 + (6\alpha + 5/16)s \end{aligned}, \quad (7)$$

$$\begin{aligned} r(s-2) &= (5/16 - 11\alpha)s^5 + (22\alpha - 5/16)s^4 \\ &- 6\alpha s^3 - 4\alpha s^2 - \alpha s \end{aligned}, \quad (8)$$

$$r(s-3) = -\alpha s^5 + \alpha s^4. \quad (9)$$

Substituting equations (4) – (9) in (3) the interpolation function is written in the form:

$$g(x)=\begin{pmatrix}21/16\cdot c_{j+1}-5/16\cdot c_{j-1}-21/16\cdot c_j\\+5/16\cdot c_{j+2}+10\alpha c_j+11\alpha c_{j-1}+\alpha c_{j-2}\\-10\alpha c_{j+1}-11\alpha c_{j+2}-\alpha c_{j+3}\end{pmatrix}s^5$$

$$+\begin{pmatrix}45/16\cdot c_j+5/4\cdot c_{j-1}-15/4\cdot c_{j+1}\\-5/16\cdot c_{j+2}-18\alpha c_j-33\alpha c_{j-1}\\-4\alpha c_{j-2}+32\alpha c_{j+1}+22\alpha c_{j+2}+\alpha c_{j+3}\end{pmatrix}s^4$$

$$+\begin{pmatrix}15/8\cdot c_{j+1}-15/8\cdot c_{j-1}+28\alpha c_{j-1}\\+6\alpha c_{j-2}-28\alpha c_{j+1}-6\alpha c_{j+2}\end{pmatrix}s^3 \tag{10}$$

$$+\begin{pmatrix}5/4\cdot c_{j-1}-5/2\cdot c_j+5/4\cdot c_{j+1}\\+8\alpha c_j-4\alpha c_{j-2}-4\alpha c_{j+2}\end{pmatrix}s^2$$

$$+\begin{pmatrix}5/16\cdot c_{j+1}-5/16\cdot c_{j-1}-6\alpha c_{j-1}\\+\alpha c_{j-2}+6\alpha c_{j+1}-\alpha c_{j+2}\end{pmatrix}s$$

$$+c_j+\mathrm{O}(h^5) .$$

Assuming that the function $f(x)$ has at least five continuous derivatives in the interval $(x_j,\ x_j+1)$, then, by applying Taylor's theorem, the value of the function in x_j+1 is calculated. With the earlier condition on the equality of the interpolation function g with the function f in the k-th interpolation nodes, the coefficients c from (3) are written in the form:

$$\begin{aligned}c_{j+1}=f(x_{j+1})&=1/4\cdot h^4f^{(4)}(x_j)\\&+1/6\cdot h^3f^{(3)}(x_j)+1/2\cdot h^2f^{(2)}(x_j),\\&+hf^{(1)}(x_j)+f(x_j)+\mathrm{O}(h^5)\end{aligned} \tag{11}$$

$$\begin{aligned}c_{j+2}=f(x_{j+2})&=2/3\cdot h^4f^{(4)}(x_j)\\&+4/3\cdot h^3f^{(3)}(x_j)+2h^2f^{(2)}(x_j),\\&+2hf^{(1)}(x_j)+f(x_j)+\mathrm{O}(h^5)\end{aligned} \tag{12}$$

$$\begin{aligned}c_{j+3}=f(x_{j+3})&=27/8\cdot h^4f^{(4)}(x_j)\\&+9/2\cdot h^3f^{(3)}(x_j)+9/2\cdot h^2f^{(2)}(x_j),\\&+3hf^{(1)}(x_j)+f(x_j)+\mathrm{O}(h^5)\end{aligned} \tag{13}$$

$$\begin{aligned}c_{j-1}=f(x_{j-1})&=1/24\cdot h^4f^{(4)}(x_j)\\&-1/6\cdot h^3f^{(3)}(x_j)+1/2\cdot h^2f^{(2)}(x_j),\\&-hf^{(1)}(x_j)+f(x_j)+\mathrm{O}(h^5)\end{aligned} \tag{14}$$

$$\begin{aligned}c_{j-2}=f(x_{j-2})&=2/3\cdot h^4f^{(4)}(x_j)\\&-4/3\cdot h^3f^{(3)}(x_j)+2h^2f^{(2)}(x_j)-2hf^{(1)}(x_j).\\&+f^{(1)}(x_j)+\mathrm{O}(h^5)\end{aligned} \tag{15}$$

After substituting eq. (11) – (15) in eq. (10), the convolutional interpolation function is written in the form:

$$g(x)=a_5s^5+a_4s^4+a_3s^3+a_2s^2+a_1s+a_0+\mathrm{O}(h^5). \tag{16}$$

where are:

$$a_5=\begin{pmatrix}9/8\cdot h^2f^{(2)}(x_j)+11/16\cdot h^3f^{(3)}(x_j)\\+1/4\cdot h^4f^{(4)}(x_j)+9/4\cdot hf^{(1)}(x_j)\\-48\alpha hf^{(1)}(x_j)-24\alpha h^2f^{(2)}(x_j)\\-24\alpha h^3f^{(3)}(x_j)-10\alpha h^4f^{(4)}(x_j)\end{pmatrix},$$

$$a_4=\begin{pmatrix}120\alpha hf^{(1)}(x_j)-5/4\cdot h^3f^{(3)}(x_j)\\-5/16\cdot h^4f^{(4)}(x_j)-45/8\cdot hf^{(1)}(x_j)\\-15/8\cdot h^2f^{(2)}(x_j)+40\alpha h^2f^{(2)}(x_j)\\+50\alpha h^3f^{(3)}(x_j)+46/3\cdot\alpha h^4f^{(4)}(x_j)\end{pmatrix},$$

$$a_3=\begin{pmatrix}5/8\cdot h^3f^{(3)}(x_j)+15/4\cdot hf^{(1)}(x_j)\\-80\alpha hf^{(1)}(x_j)-76/3\cdot h^3f^{(3)}(x_j)\end{pmatrix},$$

$$a_2=\begin{pmatrix}5/4\cdot h^2f^{(2)}(x_j)+5/4\cdot h^4f^{(4)}(x_j)\\-16\alpha h^2f^{(2)}(x_j)-16/3\cdot\alpha h^4f^{(4)}(x_j)\end{pmatrix},$$

$$a_1=\begin{pmatrix}5/48\cdot h^3f^{(3)}(x_j)+5/8\cdot hf^{(1)}(x_j)\\+8\alpha hf^{(1)}(x_j)-2/3\cdot h^3f^{(3)}(x_j)\end{pmatrix},$$

and

$$a_0=f(x_j).$$

The expansion of the function f into Taylor series is obtained:

$$\begin{aligned}f(x)&=1/24\cdot h^4f^{(4)}(x_j)s^4+1/6\cdot h^3f^{(3)}(x_j)s^3\\&+1/2\cdot h^2f^{(2)}(x_j)s^2+hf^{(1)}(x_j)s+f(x_j)+\mathrm{O}(h^5)\end{aligned}. \tag{17}$$

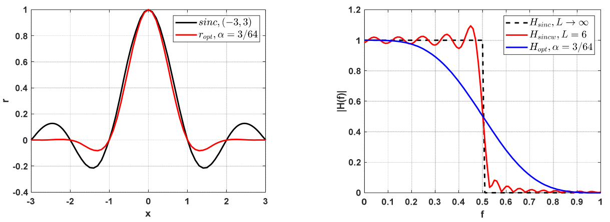
The interpolation error is:

$$\begin{aligned}e(x)&=f(x)-g(x)\\&=\begin{pmatrix}1/4\cdot A\cdot h^4f^{(4)}(x_j)+1/16\cdot B\cdot h^3f^{(3)}(x_j)\\+3/8\cdot Ch^2f^{(2)}(x_j)+3/4\cdot C\cdot hf^{(1)}(x_j)\end{pmatrix}s^5\\&+\begin{pmatrix}-1/48\cdot D\cdot h^4f^{(4)}(x_j)-5/4\cdot E\cdot h^3f^{(3)}(x_j)\\-5/8\cdot C\cdot h^2f^{(2)}(x_j)-15/8\cdot C\cdot hf^{(1)}(x_j)\end{pmatrix}s^4\\&+\left(1/24\cdot F\cdot h^3f^{(3)}(x_j)+5/4\cdot C\cdot hf^{(1)}(x_j)\right)s^3\\&+\left(1/48\cdot G\cdot h^4f^{(4)}(x_j)+1/4\cdot C\cdot h^2f^{(2)}(x_j)\right)s^2\\&+\begin{pmatrix}1/48\cdot Hh^3f^{(3)}(x_j)\\-1/8\cdot C\cdot hf^{(1)}(x_j)\end{pmatrix}s+\mathrm{O}(h^5)\end{aligned}, \tag{18}$$

where are the coefficients $A=40\alpha$ -1, $B=384\alpha$ -11, $C=64\alpha$ -3, $D=736\alpha$ -17, $E=40\alpha$ -1, $F=608\alpha$ -11, $G=256\alpha$ -5, and $H=32\alpha$ -5. In order to minimize the interpolation error e, it is necessary that the coefficients A, ... , H are as small as possible, ideally equal to zero. This can only be achieved if the first five terms of the Taylor series expansion for f and g are mutually equal, that is, if their difference is as small as possible. However, because the coefficients have a common variable, which is the parameter α, it is not possible to achieve a unique solution. By equating the coefficients A, ... , H to zero, and marking the parameters α in the index with the name of the coefficient, it is obtained that: $\alpha_A = 1/40$, $\alpha_B = 11/384$, $\alpha_C = 3/64$, $\alpha_D = 3/736$, $\alpha_E = 1/40$, $\alpha_F = 11/608$, $\alpha_G = 5/256$ and $\alpha_H = 5/32$.

It is observed that, out of fourteen coefficients (eq. 18), the term $C = 64\alpha - 3$ appears 7 times. In the final calculation of the optimal parameter α_{opt}, two important facts should be analyzed: a). Mean value of the parameters is $\overline{\alpha} = 1/14 \cdot \sum_{z=A}^{H} \alpha_z = 0.0499$. The smallest difference between $\overline{\alpha}$ and all other parameters α_A, ..., α_H is in relation to α_C. Their difference is $\Delta\alpha_C = \overline{\alpha} - \alpha_C = 0.0499 - 3/64 = 0.0030$. b) In the paper [15] optimization of the kernel was performed in the spectral domain with the criterion of minimizing the ripple of the spectral characteristics. The minimum ripple of the spectral characteristic is achieved with $\alpha_{opt} = 3/64$. Taking into account the mentioned facts, the conclusion is reached that the optimal value of the kernel parameter of the interpolation kernel is $\alpha_{opt} = \alpha_C = 3/64$.

In fig. 1.a shows the time forms of: a) the ideal interpolation kernel $sinc$, and b) the fifth-order polynomial kernel, r_{opt}, with the optimal parameter $\alpha_{opt} = \alpha_C = 3/64$, on the interval (-3, 3). In fig. 1.b shows the spectral characteristics of: a) ideal interpolation kernel H_{sinc} (length $L \to \infty$), b) windowized ideal kernel H_{sincw} length $L = 6$, and c) optimized fifth-order kernel H_{opt} with $\alpha_{opt} = 3/64$.



a) b)

Fig.1. a) Time forms of: the ideal interpolation kernel *sinc*, and the fifth-order polynomial kernel, on the interval (-3, 3); b) Spectral characteristics of: ideal interpolation kernel H_{sinc} (length $L \to \infty$), windowized ideal kernel H_{sincw} length $L = 6$, and optimized fifth-order kernel H_{opt} with $\alpha_{opt} = 3/64$.

Experimental results and analysis

Experiment

In order to verify the precision of interpolation, an experiment was carried out. The fifth-order interpolation 1P kernel, with implemented all analyzed kernel parameters (α_A, α_B, α_C, α_D, α_E, α_F, α_G, α_H), was used. The interpolation error $e(x) = f(x) - g(x)$, for each interpolation, was calculated. After that, the mean squared error MSE, which is defined as $MSE = 1/N \cdot \sum_{n=1}^{N} e^2$, where N is the number of interpolation points, is determined. MSE was used as a measure of interpolation precision. A smaller value of MSE indicates a higher precision of interpolations.

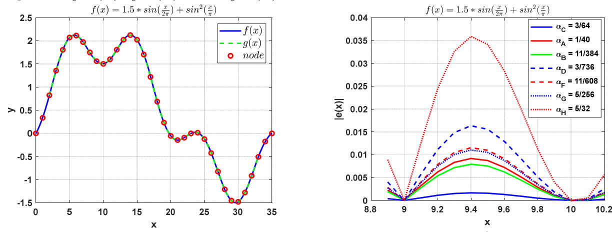
The interpolation was carried out over the test signals $f_1(x)$, $f_2(x)$ and $f_3(x)$, which are defined as:

$$f_1(x) = 1.5 \cdot sin(\frac{x}{2\pi}) + sin^2(\frac{x}{\pi}) , \tag{19}$$

$$f_2(x) = 10^{-3} \cdot (x-10) \cdot (x-15) \cdot (x-35) \cdot sin(\frac{x}{\pi}) , \tag{20}$$

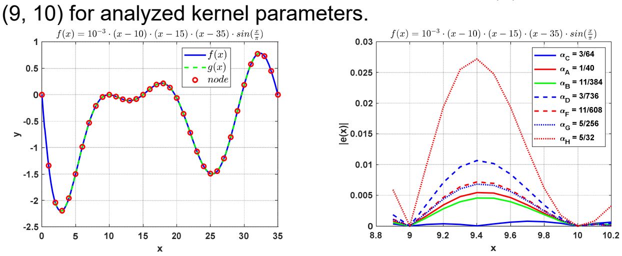
$$f_3(x) = e^{-\frac{x}{2\cdot\pi}} \cdot sin(4 \cdot \frac{x}{\pi}) . \tag{21}$$

Test signals $f(x)$, interpolation functions $g(x)$, and interpolation nodes are shown in: fig. 2.a ($f_1(x)$), fig. 3.a ($f_2(x)$), and fig. 4.a ($f_3(x)$). The absolute interpolation error $|e|$, for the tested kernel parameters α, on the segment (9, 10), are shown in: fig 2.b, fig. 3.b, and fig. 4.b. In tbl. 1 shows MSE for all analyzed kernel parameters, for test signals $f_1(x)$, $f_2(x)$ and $f_3(x)$.



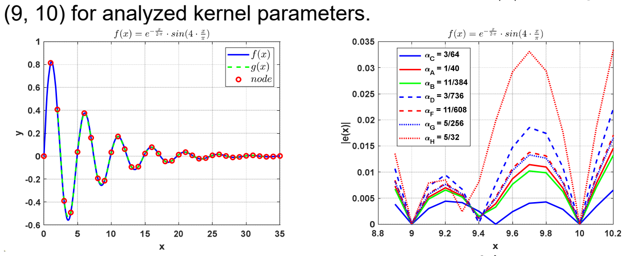
a) b)

Fig.2. a) Interpolated signal $f_1(x)$, interpolation function $g_1(x)$ and interpolation nodes n. Absolute interpolation error $|e|$ on segment (9, 10) for analyzed kernel parameters.



a) b)

Fig.3. a) Interpolated signal $f_2(x)$, interpolation function $g_2(x)$ and interpolation nodes n. Absolute interpolation error $|e|$ on segment (9, 10) for analyzed kernel parameters.



a) b)

Fig.4. a) Interpolated signal $f_3(x)$, interpolation function $g_3(x)$ and interpolation nodes n. Absolute interpolation error $|e|$ on segment (9, 10) for analyzed kernel parameters.

Table 1. *MSE* for the test signal $f_1(x)$, $f_2(x)$, and $f_3(x)$, depending on the analyzed kernel parameters.

MSE	$f_1(x)$	$f_2(x)$	$f_3(x)$
MSE_C	$1.0892\cdot10^{-06}$	$5.0582\cdot10^{-07}$	$3.7074\cdot10^{-05}$
MSE_A	$6.0202\cdot10^{-05}$	$5.6243\cdot10^{-05}$	$1.6468\cdot10_{-04}$
MSE_B	$4.3937\cdot10^{-05}$	$4.0200\cdot10^{-05}$	$1.3677\cdot10^{-04}$
MSE_D	$2.0316\cdot10^{-04}$	$2.0056\cdot10^{-04}$	$3.7626\cdot10^{-04}$
MSE_F	$9.8055\cdot10^{-05}$	$9.4050\cdot10^{-05}$	$2.2485\cdot10^{-04}$
MSE_G	$8.9409\cdot10^{-05}$	$8.5373\cdot10^{-05}$	$2.1153\cdot10^{-04}$
MSE_H	0.0011	0.0012	$8.3395\cdot10^{-04}$

Analysis of results

Based on the results, which are graphically shown in fig. 2 - fig. 4, and in table 1, it can be concluded that the interpolation error MSE, for kernels with analyzed kernel parameters (α_A , α_B, , α_D, α_E, α_F, α_G and α_H), in relation to the kernel with the optimal parameter ($\alpha_{opt} = \alpha_C$), for test signals ($f_1(x)$, $f_2(x)$ and $f_3(x)$) interpolation, is greater:

a) $f_1 \Rightarrow$.. MSE_A / MSE_C = $6.0202\cdot10^{-05}$ / $1.0892\cdot10^{-06}$ = 55.3, MSE_D / MSE_C = $4.3937\cdot10^{-05}$ / $1.0892\cdot10^{-06}$ = 40.3, MSE_D / MSE_C = $2.0316\cdot10^{-04}$ / $1.0892\cdot10^{-06}$ = 186.5, MSE_F / MSE_C = $9.8055\cdot10^{-05}$ / $1.0892\cdot10^{-06}$ = 90.0, MSE_G / MSE_C =

$8.9409 \cdot 10^{-05}$ / $1.0892 \cdot 10^{-06} = 082.1$, MSE_H / $MSE_C = 0.0011$ / $1.0892 \cdot 10^{-06} = 1009.9$ times, respectively,

b) $f_2 \Rightarrow$.. MSE_A / $MSE_C = 5.6243 \cdot 10^{-05}$ / $5.0582 \cdot 10^{-07}$ = 111.2, MSE_B / $MSE_C = 4.0200 \cdot 10^{-05}$ / $5.0582 \cdot 10^{-07} = 079.5$, MSE_D / $MSE_C = 2.0056 \cdot 10^{-04}$ / $5.0582 \cdot 10^{-07} = 396.5$, MSE_F / $MSE_C = 9.4050 \cdot 10^{-05}$ / $5.0582 \cdot 10^{-07} = 185.9$, MSE_G / $MSE_C = 8.5373 \cdot 10^{-05}$ / $5.0582 \cdot 10^{-07} = 168.8$, MSE_H / $MSE_C = 0.0012$ / $5.0582 \cdot 10^{-07} = 2372.4$ times, respectively,

c) $f_3 \Rightarrow$..MSE_A / $MSE_C = 1.6468 \cdot 10^{-04}$ / $3.7074 \cdot 10^{-05}$ = 4.4419, MSE_B / $MSE_C = 1.3677 \cdot 10^{-04}$ / $3.7074 \cdot 10^{-05}$ = 3.6891, MSE_D / $MSE_C = 3.7626 \cdot 10^{-04}$ / $3.7074 \cdot 10^{-05}$ =10.1489, MSE_F / $MSE_C = 2.2485 \cdot 10^{-04}$ / $3.7074 \cdot 10^{-05}$ = 6.0649, MSE_G / $MSE_C = 2.1153 \cdot 10^{-04}$ / $3.7074 \cdot 10^{-05}$ = 5.7056, MSE_H / MSE_C = = $8.3395 \cdot 10^{-04}$ / $3.7074 \cdot 10^{-05}$ = 22.4942 times, respectively,

Taking into account the theoretical analysis, which was realized in the time domain, as well as the experimental results, it is unequivocally confirmed that the optimal value of the kernel parameter $\alpha_{opt} = 3/64$ is well chosen.

Conclusion

The paper describes the optimization process of the fifth-order polynomial one-parameter interpolation kernel. The optimization of the 1P kernel involved the selection of the optimal value of the kernel parameter α_{opt}. The optimization was realized by minimization of the interpolation error e in the time domain. First, the fifth-order 1P kernel r, which is defined on the interval (-3, 3), is described. Then, by applying convolutional interpolation between the interpolated function f and the 1P kernel r, the interpolation function g is determined. Interpolated function f and interpolation function g in interpolation nodes are equal. After that, the interpolation error e, in the interval $x_j < x < x_{j+1}$, is determined. With the condition that the function f has at least five continuous derivatives in the interval (x_j, x_{j+1}), the interpolation error e is developed in the Taylor series up to the fifth term. According to the minimization criterion, it is necessary to agree well functions f and g, up to the fifth terms. By minimizing the first five terms of the Taylor series of the interpolation error e, the optimal value of the kernel parameter can be calculated. However, the minimization process does not lead to a unique solution for α. Each of the five terms of the Taylor series has a coefficient, which depends on several members, which are represented in the form ($a \cdot \alpha + b$). All five coefficients have a total of fourteen such members, of which the form ($64 \cdot \alpha$ - 3) appears seven times. Therefore, the value $\alpha = 3/64$ is imposed as the optimal value. The mean value $\overline{\alpha}$ = 0.0499 is the closest value of $\alpha = 3/64$. In addition, $\alpha = 3/64$ is equal to the value of the optimal kernel parameter, which was determined by optimization in the spectral domain, when the optimization criterion was the minimization of the ripple of the spectral characteristic. Therefore, after detailed analysis, the optimal value is $\alpha_{opt} = 3/64$ is proposed. By using an experiment, the verification of the proposed optimal value of the kernel parameter was realized. Three test signals, with complex time forms, were created. Each test signal was interpolated by convolutional interpolation, where an interpolation kernel, in which all the analyzed values of the kernel parameter α were individually implemented, was used. For each interpolation, the interpolation errors were calculated, and, based on them, the *MSEs*, which were used for comparative analysis, were formed. A detailed comparative analysis showed that the precision of interpolation, with the kernel parameter $\alpha = 3/64$, compared to the precision of the other analyzed values of the kernel parameters, is the highest. In this way, the suggested optimal values were verified.

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