Michał K. URBAŃSKI, Kinga WÓJCICKA, Karol BOLEK, Paweł WÓJCICKI, Maksymilian BIAŁAS

OPEN ACCESS

DOI: 10.15199/48.2025.05.34

Fuzzy approach to analyze measurement uncertainty on the example of magnetic field measurement

Analiza niepewności metodą zbiorów rozmytych na przykładzie pomiaru zmiennego pola magnetycznego

Abstract. *The paper shows an example of the application of fuzzy set theory to calculate the fuzzy uncertainty in the measurement of a alternating magnetic fields in a system using a induction sensor.*

Streszczenie. *W pracy pokazano przykład zastosowania teorii zbiorów rozmytych do wyliczenia niepewności rozmytej w pomiarze zmiennego pola magnetycznego w układzie wykorzystującym czujnik indukcyjny.*

Keywords: fuzzy sets, fuzzy variable, uncertainty, theory of measurements, probability to possibility transformation
Słowa kluczowe: zbiory rozmyte, zmienna rozmyta, niepewność rozszerzona, niepewność rozmyta

Introduction

In the papers [2, 9, 5, 6], a description of measurement uncertainties in the fuzzy sets model was proposed. In this paper, we will demonstrate the application of fuzzy set theory using the example of determining the measurement uncertainty of the magnetic induction of a alternating magnetic field with an instrument built in our laboratory. The constructed instrument is not a high-precision device but a measurement system built from scratch, in which the components of uncertainty can be controlled. It is assumed that the standard of the magnetic field is a solenoid powered by a sinusoidal voltage generator with known voltage, and the system for measuring the magnetic field is an inductive type.

In this paper [1, 3, 5] we assume that the error δX is the difference between the measurement result X and the true value x_0:

$$X = x_0 + \delta X \tag{1}$$

In the framework of probabilistic model the measurement result X and the error δX can be represented by a random variables and in fuzzy model – by fuzzy variables. We assume that uncertainty $U(X)$ is a measure of error and is equal to the parameter of the distribution (probabilistic of fuzzy) of variable X. If the true value is a real number, then according to (1), adding a real number to a random or fuzzy variable does not change the uncertainty:

$$U(X) = U(\delta X) \tag{2}$$

We do not find such definitions in the „Guide to the Expression of Uncertainty in Measurement" [1], but we assume that they are necessary to explain in the language of probability that the standard uncertainty is the standard deviation and expanded uncertainty is equal to the radius of the confidence interval. In the fuzzy sets model, fuzzy uncertainty is the radius of the section of the fuzzy set characterizing the error at a certain level of possibility α, which corresponds to expanded uncertainty at the level $p(\alpha)$[1].

[1] The definition of standard uncertainty requires an additivity of measure, but in fuzzy sets, the possibility measure is maxitive, therefore, it is not possible to define a measure adequate to the standard deviation.

Equation (1) describes both single sample measurements and series of measurements. A single sample measurement can be described by an interval model, where X and δX are intervals with radii equal to the maximum permissible errors. The interval model is equivalent to the assumption that both the measurement result X and the error δX are fuzzy variables with a uniform fuzzy distribution. In the framework of a probabilistic model, X and δX are represented by random variables, and in a fuzzy model – by fuzzy variables. From the definition, the distribution of a fuzzy variable is a fuzzy set, therefore equation (1) can be written for fuzzy sets. Writing this equation for fuzzy sets is equivalent to the equation for adding distributions of random variables.

In the Guide, the division into two methods of determining uncertainty is given without justification. In the language of probability theory, this means two methods of estimating the probability distribution:

A – *a posteriori*, which involves determining the probability distribution from empirical data using the frequentist interpretation of probability,

B – *a priori*, which involves estimating the probability distribution based on expert analysis using knowledge of the measurement method and the instruments used.

Therefore we assumed that the total error is the sum of components A and B: $\delta X = \delta X_A + \delta X_B$. The formulas for combining uncertainty are based on the rules for determining the distribution of the sum of fuzzy variables, which means adding fuzzy sets according to Zadeh's extension principle.

In the probabilistic model, the standard uncertainty $u(X)$ is equal to the estimator s of the standard deviation of the error: $u(X) = u(x_0 + \delta X) = s(\delta X)$ The expanded uncertainty is equal to the radius of the confidence interval $I_p(\Phi(X))$ of the estimator $\Phi(X)$ of the measured quantity (measurand) at the level p:

$$U_p(X) = U_p(x_0 + \delta X) = \mathrm{rad}\,(I_p(\delta X)) \tag{3}$$

In equation (3), we assume that the confidence interval does not change when shifting the probability distribution along the X axis and that expanded uncertainty can be defined through the confidence interval[2].

In the fuzzy sets model, we will adopt the following assumptions:

1. The expanded uncertainty at level α is equal to the radius of the fuzzy set section at level α; to compare with the uncertainty in the probabilistic model, we assume that the section of the fuzzy set at level α corresponds to the radius of the confidence interval at the level[3] $p = 1 - \alpha$,
2. Method B for uncertainty assessment involves an *a posteriori* expert method; to describe such a situation Zadeh proposed the theory of fuzzy sets,
3. The formulas for combining type A and type B uncertainties result from the arithmetic of fuzzy sets,
4. If we assume that the type B component (in many cases it can be identified with the systematic error) is described by a uniform membership function within an interval, then the fuzzy combined uncertainty is the arithmetic sum of the *a posteriori* and *a priori* components [7].

Probability to possibility transformation

The basis of uncertainty analysis proposed here using the fuzzy sets method is the probability-to-possibility transformation described in [6, 5, 9, 8] for type *A* uncertainty component.

If we assume that the confidence interval I_p on the p- level is equal to the α-cut of a fuzzy set for $p = 1 - \alpha$, we obtain the relation between the fuzzy set $\bar{A}_X$ and the cumulative distribution function F_X of the random variable X.

That is, if:

(4) $$I_p = [A_X]^{1-p}$$

where I_p is defined as: $P(I_p) = p$, then the fuzzy set $\bar{A}_X$ is given by the equation ([6, 5, 9])

(5) $$\bar{A}_X(x) = \begin{cases} 2F_X(x) & for \quad x \le M(X) \\ 2 - 2F_X(x) & for \quad x > M(X) \end{cases}$$

where $M(X)$ is the median of a random variable X and F_X is the cumulative probabilistic distribution of a random variable X.

Fuzzy sets derived from this transformation describe the components of uncertainty of type A, whereas the equation (5) define a fuzzy set of type $L - R$ (for unimodal distributions).

Fuzzy approach to measurement uncertainty, error budget

Equation (1) means that all sources of measurement inaccuracies are represented by the error δX. The exact value is unknown, and the fact of the existence of errors is included in the assumption that the error is a random variable or a fuzzy variable (the interval model is a case of fuzzy sets).

The starting point for uncertainty estimation is the error budget4. The error budget describes the components of the measurement error. If the error has two components: $\delta x = \delta x_1 + \delta x_2$, then the uncertainty $U_\alpha(\delta x) = U_\alpha(\delta x_1 + \delta x_2)$ depends on the distribution of the sum of the fuzzy variables $\delta X_1 + \delta X_2$, and the sum distribution depends on the t-norm describing the interaction between these variables.

If these variables are observed as a set of data with a certain spread, then by using the probability-to-possibility transformation, we determine the appropriate membership functions using the algorithm described in the papers [6, 5, 9]. We will summarize this algorithm in the next chapter.

The distribution of the type B error component is determined analogously to postulating the probabilistic distribution. If the data from the instrument indicates that the maximal bound error is ΔX, then the distribution describing this situation is a rectangular distribution on the interval $[-\Delta X, \Delta X]$.

Calculation of membership function from measurement data

Let's assume we obtain a series of n measurement data $\{x_1, \ldots x_n\}$. The transformation described by the formula (5) in the discrete case is illustrated by the figure: at each measuring point below the median, there is a jump of value by $\frac{2}{n}$.

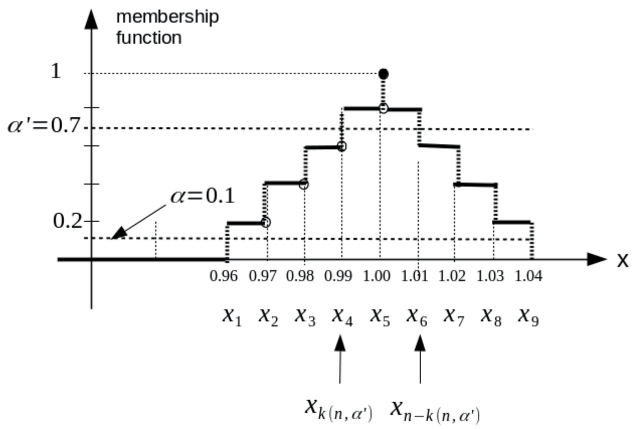

Fig. 1. Empirical estimation of fuzzy set and α-cut in the case $n = 9$

Determination of the membership function can be described by the following algorithm (fig. 1):

1. Sort the measurement data: x1 < . . . < xn.
2. For the initial value of x1, we set A ̄(x1) = 0.
3. Determine $\bar{A}(x_i)$ using the recursive formula: For even n, $i \le \frac{2}{n}$: $\bar{A}(x_i+1) = \bar{A}(xi) + \frac{2}{n}$ for $\frac{2}{n} < i <$ n: $\bar{A}(x_i+1) = \bar{A}(x_i) - \frac{2}{n}$. For odd n it is analogous.
4. For i = n we obtain: $\bar{A}(x_n) = 0$.
5. Interpolation between measuring points is done with horizontal lines, obtaining the step function.
6. Since the uncertainty is determined by the radius of the α-cut, we have:

(6) $$[\bar{A}]^\alpha = [x_{k(n,\alpha)}, x_{n-k(n,\alpha)}]$$

[2] In the GUM we find that „the confidence interval can be applied only to method A" but we assume that it can also be applied to method B; otherwise, the formula for combined uncertainty cannot be justified

[3] Another relationship between the confidence level and the degree of membership requires further research.

[4] The uncertainty budget used in metrology is a „bag" written with many statistical assumptions.

where $\{x_i\}i = {}_{1,n}$ is a sequence of sorted data, and $k(n,\alpha)$ is the sequential number of the data forming the left end-points of the α-cut (see fig. 1).

$$(7)\qquad k(n,\alpha) = \begin{cases} \left\lceil \frac{n}{2}\alpha \right\rceil & \text{for even } n \\ \left\lceil \frac{(n+1)}{2}\alpha \right\rceil & \text{for odd } n \end{cases}$$

If the measurand is defined as the average value of the data series, then the α-cut of the membership function representing the average value for a particular T-norm should be determined. In the fuzzy case, averaging results (as in the probabilistic approach) is a narrowing of the distribution. We can derive the formula providing the α-cut of the distribution $\bar{A}_{\text{Ave}}$ of the average values:

$$(8)\qquad [\bar{A}_{\text{Ave}}]^{\alpha} = \left[\text{Ave}_n(\bar{A})\right]^{\alpha} = \left[\bar{A}\right]^{\left(t^{[-1]}\left(\frac{1}{n}t(\alpha)\right)\right)}$$

This formula means that in order to determine the α-cut of the distribution of average values at level α, it is necessary to determine the α-cut of non-averaged values at level $\alpha' = t^{-1}\left(\frac{1}{n}t(\alpha)\right)$ (where t is the additive generator of the T-norm).

Measurement system

In the studied system, the measurement of the alternating magnetic field is carried out using the measurement of voltage on an air coil, i.e., by the induction method [4]. The measuring instrument is described by a calibration function that describes the relationship between the output voltage VOUT and the magnetic induction field B, which in linear approximation has the form:

$$(9)\qquad V_{OUT} = \alpha B$$

where α is calibration parameter. In order to determine the calibration parameter α of the measuring instrument, we measure, in the system presented in fig. 2, the relation between the output voltage V_{OUT} and the generator voltage V_G. Assuming the linearity of the system, the relationship between the output voltage and the generator voltage of the calibration system (fig. 2) has the form:

$$(10)\qquad V_{OUT} = aV_G$$

The coefficient α is determined from the measurement data of the relationship between the output voltage U_{OUT} and the generator voltage U_G in the measurement system of the calibration setup in fig. 2.

The standard used for calibration is a long solenoid. The magnetic field B generated by the generator voltage V_G is theoretically described by Ampere's law and approximately defines a linear relationship:

$$(11)\qquad B = \beta V_G$$

where β is a coefficient resulting from the laws of electrodynamics. Using the long solenoid approximation, the magnetic field B is equal to:

$$(12)\qquad B = \mu_0 N\frac{I}{l} = \mu_0 n_0 I$$

where N is the number of turns, I is the electric current, l is the solenoid length, and n_0 = $n_0 = \frac{N}{l}$ is the number of turns per unit length.

Since the input circuit consists of inductance L and resistance R, the RMS value of current I is given by I = √ VG $\frac{V_G}{\sqrt{(\omega L_0)^2+R_L^2}}$, which means:

$$(13)\qquad \beta = \mu_0\frac{n_0}{\sqrt{(\omega L_0)^2 + R_L^2}}$$

The voltage on the measuring coil is: $U = \frac{d\Phi_b}{dt}$ The signal amplification system from the measuring coil works as a current-to-voltage converter. The measuring coil is a radio antenna coil for medium waves with an inductance of L_1 = 665 ± 0.005μH, n_1 turns and a resistance of R_1 = 18.68 ± 0.02Ω. The characteristic frequency of the coil $f_L = \frac{1}{2\pi}\frac{R_1}{L_1}$ = 4.48 kHz. For $f > f_L$, the formula for the output voltage as a function of the magnetic flux of the system is:

$$(14)\qquad V_{OUT} = \Phi n_1\frac{R_f}{L_1} = BS_1 n_1\frac{R_f}{L_1}$$

where R_f is the feedback resistance, and S1 is the effective area of the measuring coil. This equation is linear (from (18)) with the slope coefficient $\alpha = S_1 n_1\frac{R_f}{L_1}$ (9) . For more accurate measurements, the influence of frequency on the system gain should be considered.

Substitute equation (11) into equation (10), we obtain: $V_{OUT} = \alpha\beta V_G$ Therefore, the coefficient a of equation (10) is a = αβ, so the coefficient α of the field meter calibration curve is:

$$(15)\qquad \alpha = \frac{a}{\beta}$$

The measuring coil signal amplifier

To derive equation (14), let us consider Kirchhoff's laws for the equivalent circuit shown in figure 2. The voltage induced in the measuring coil L is equal to the time derivative of the magnetic flux, which in complex numbers is represented by:

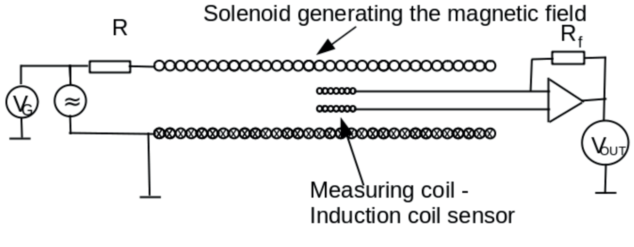

Fig. 2. Measuring circuit for calibration of induction field meter

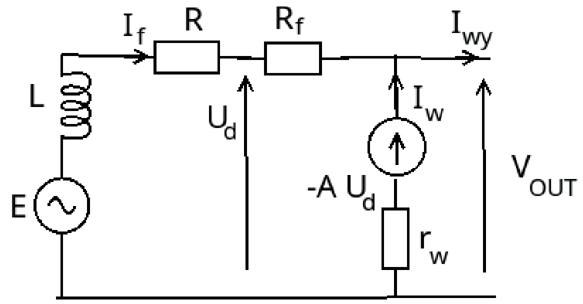

Fig. 3. Equivalence scheme, E – measuring coil voltage

(16) $E = jn_1\omega\Phi = j\omega n_1 S_1 B$

where ω is the angular frequency, Φ is the magnetic flux, S_1 is the effective surface area of the coil, and n_1 is the number of coil turns. The equations describing the system take the form (with the assumption $|I_w R_w| << |AU_d|$):

$$E - U_d = I_f(R + j\omega) \tag{17}$$
$$U_d - V_{OUT} = R_f I_f \text{ and } V_{OUT} = -AU_d$$

where: A – open loop gain of operational amplifier. After some manipulations we obtain for $|A| >> 1$:

$$V_{OUT} = -jn_1\omega\Phi\frac{R_f}{R + j\omega L_1} \tag{18}$$

In the discussed system, the characteristic frequency is lower than the one used for measurements and standardization, so an approximate formula (14) for the effective values can be used. At low frequencies, to make the amplification independent of the frequency, the amplifier circuit needs to be modified by using an integrator circuit.

Fuzzy uncertainty analysis

The system is designed so that within a certain frequency range, the output voltage is proportional to the magnetic induction vector. The relative error of the proportionality coefficient α, according to the formula (15), is:

$$\frac{\delta\alpha}{\alpha} = \frac{\delta a}{a} - \frac{\delta\beta}{\beta} \tag{19}$$

The coefficient a is determined by fitting the straight-line equation to the measurement data of the calibration system, and it is necessary to determine the membership function for the slope coefficient resulting from the fuzzy method of linear fitting. Unfortunately, in the literature, we did not find suitable equations using the fuzzy metric consistent with our proposed fuzzy order [10]. Therefore, we used the least squares fitting method, obtaining a = 6.805 and a standard deviation $\sigma(a)$ = 0.081. We take $\Delta a = 2\sigma(a)$ as the fuzzy uncertainty. A consistent fuzzy fitting method should be developed using the fuzzy metric.

The coefficient β describes the theoretical dependence of field induction on the generator voltage. According to equation (13), the budget of relative errors is:

$$\frac{\delta\beta}{\beta} = \frac{\delta n_0}{n_0} + \frac{\delta L}{L} + \frac{\delta\omega}{\omega}$$

The value of n_0 was determined by measuring the length occupied by N = 40 turns, resulting in l = 40 mm ± 0.4 mm; therefore: $\frac{\delta n_0}{n_0} = \frac{\delta l_0}{l_0}$ = 0.01. The inductance L was measured using an inductance meter, yielding L = 1.1 mH ± 0.01 mH, so $\frac{\delta L}{L}$ = 0.01. The frequencies were read from the generator, and therefore frequency uncertainty is much lower than for the other components, so it does not occur in these formulas.

According to the principles resulting from the addition of fuzzy sets, if the components of errors are described by rectangular fuzzy sets, then the cross-section of the fuzzy set that is the sum of these sets is the algebraic sum of the cross-sections, and the error propagation principle is consistent with interval arithmetic:

$$\frac{\Delta\alpha}{\alpha} = \frac{2\sigma(a)}{a} + \frac{\Delta n_0}{n_0} + \frac{\Delta L}{L} \simeq 0,03 \tag{20}$$

The value of the magnetic field induction is B = $\frac{V_{OUT}}{\alpha}$. The source of the measured field was an air coil with unknown parameters (which are not essential for this measurement), powered by a generator producing a sinusoidal signal at a frequency of f = 10 kHz.

Using a digital oscilloscope, a series of RMS voltage measurements were taken for a single frequency and a single generator voltage. Variability in voltage readings was observed, likely caused by field disturbances. By applying the algorithm shown in Figure 1 (and below the picture) for the series of RMS voltages, we obtained the membership function depicted in the figure. The membership function is not symmetrical and does not correspond to a Gaussian distribution. Calculations using this method do not require any assumptions about the shape of distribution but do require assumptions about the t-norm, which corresponds to the copula function. Uncertainty calculations using the probabilistic method are usually performed under the assumption that the distributions are normal and with hidden assumptions about the correlation of random variables describing the components of errors in the error budget. It is also assumed that there are no correlations within the data set on the basis of which type A uncertainties are determined. Therefore one can assumed that the standard deviation of the mean σ(XAV) is equal to $\sigma(X_{AV}) = \sigma(X_{AV}) = \frac{\sigma(X)}{\sqrt{N}}$, which requires independence of events.

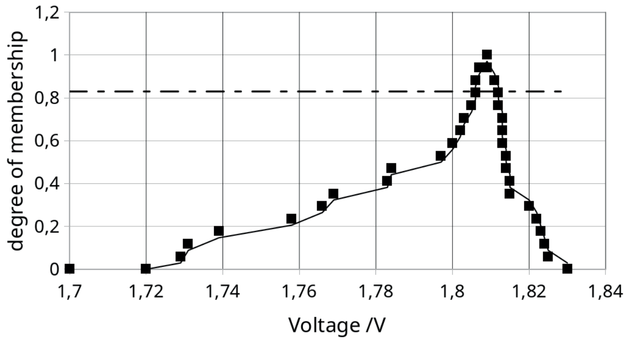

Fig. 4. The membership function of the output voltage, the horizontal dashed line at the level α = 0.83 indicates the cut at the level α' = $t^{[-1]}\left(\frac{1}{n}t(\alpha)\right)$

As seen in figure 4, the α-cut of the membership function of the average value is the interval [1.806, 1.812]. The fuzzy uncertainty is equal to the radius of this interval; therefore, ΔU = 0.003 V, and the relative uncertainty is: $\frac{\Delta U}{U}$ = 0.0017. In this experiment, voltage measurement errors are the smallest component and can be neglected. Still, we will write down the general formulas for the uncertainty of magnetic field measurement:

$$\frac{\Delta B}{B} = \frac{\Delta U}{U} + \frac{\Delta\alpha}{\alpha} = \frac{\Delta U}{U} + \frac{2\sigma(a)}{a} + \frac{\Delta n_0}{n_0} + \frac{\Delta L}{L} \tag{21}$$

The voltage measurement error component is type A, while the others are type B. In the fuzzy sets model, combining type A and type B uncertainties is arithmetic regardless of the t-norm. This is a significant difference compared to the GUM algorithm, which treats type A and B error components as independent random variables, which raises considerable concerns. Ultimately, the relative uncertainty of the field measurement in this experiment is 0.03, whereas the GUM method calculations yield approximately 0.04. It cannot be said that the fuzzy method is better because it gives a smaller value, due both methods provide different interpretations.

Conclusions

In the framework of probabilistic model the measurement is represented by sampling from some sample space with some probability distribution. Sample space is a set of possible measurement results. From the point of view of fuzzy model the measurement is based on comparison, as a result of which the degree of matching of the standards to the properties of the measured objects is determined. The uncertainty in probabilistic model describes the spread of measurement series. In a fuzzy model, uncertainty describes the degrees to which a set of patterns reflects the characteristics of the object being measured.

Authors: *dr hab. inż. Michał Urbańnski, email: michal.urbanski@pw.edu.pl, mgr in˙z. Kinga Wójcicka, email: kinga.wojcicka.dokt@pw.edu.pl, mgr in˙z. Karol Bolek, email: karol.bolek@pw.edu.pl, Maksymilian Białas, email:Maksymilian Białas <maks.bialas@gmail.com> Faculty of Physics, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warszawa, Poland, dr Paweł Wójcicki, email: Pawel.Wojcicki@pw.edu.pl, Faculty of Mathematics and Informational Sciences, Warsaw University of Technology.*

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