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Factor Graph-Based Trajectory Estimation for Radar-Based Missile Tracking

Estymacja trajektorii oparta na grafach czynnikowych dla radarowego śledzenia pocisków

***Abstract**. This study explores the application of factor graphs in missile trajectory tracking, presenting an alternative to traditional methods like the Extended Kalman Filter (EKF). By modeling the entire trajectory within a batch optimization framework, factor graphs enable more accurate estimates by considering all available information at once. The results show that factor graphs outperform EKF across multiple scenarios. These findings highlight the potential of factor graphs for precise trajectory estimation and their broader applicability to other tracking systems.*

***Streszczenie.** Publikacja zawiera analizę możliwości zastosowania grafów czynnikowych w śledzeniu trajektorii pocisków, stanowiąc alternatywę dla tradycyjnych metod, takich jak rozszerzony filtr Kalmana. Grafy dwudzielne pozwalają na modelowanie całej trajektorii jako problemu optymalizacji, umożliwiając dokładniejsze oszacowania przez jednoczesne uwzględnienie wszystkich dostępnych informacji. Wyniki pokazują, że grafy czynnikowe umożliwiają dokładniejszą estymację niż rozszerzony filtr Kalmana w różnych scenariuszach. Wyniki te podkreślają potencjał grafów w precyzyjnej estymacji trajektorii oraz ich szersze zastosowanie w innych systemach śledzenia.*

Keywords: factor graph, tracking, trajectory estimation, radar
Słowa kluczowe: graf czynnikowy, śledzenie, estymacja trajektorii, radar.

Introduction

Modern radar systems play a vital role in tracking and predicting the trajectories of missiles and other projectiles. The dynamics of missile trajectories can be influenced by gravitational forces, aerodynamic drag, and external disturbances, such as wind. Accurate estimation of these trajectories is essential for both civilian and defence applications [1].

Traditional trajectory estimation methods, such as the Kalman filter (KF), are rooted in the assumption of a Markov process, where each state depends only on its immediate predecessor. The KF employs recursive updates to estimate state variables based on measurements. This method is computationally efficient; however, its accuracy is constrained by the incomplete incorporation of dynamic constraints and measurement limitations in each filter iteration [2].

An alternative formulation of the tracking problem involves the use of a factor graph, where the nodes correspond to the poses of the object at different time points and the edges represent the constraints between these poses. Factor graphs provide a batch optimisation framework for smoothing problems by modelling the entire trajectory through the simultaneous consideration of all measurements and state transitions, as opposed to the recursive updates used in filtering approaches [3]. This smoothing-based formulation allows factor graphs to more effectively handle nonlinear relationships and sparse observations.

Factor graphs were first introduced by Kschischang et al. in [4] to describe the factorisation of functions in probabilistic graphical models. These graphs were later utilized in the sum-product algorithm for efficient marginalization. In robotics, many approaches extended the application of factor graphs to solve Simultaneous Localization and Mapping (SLAM) problems efficiently [5, 6, 7]. Subsequent advancements included techniques for handling non-linear constraints and applying factor graphs to navigation systems [6, 8].

Research by Kaess et al. [9] demonstrated that factor graphs, coupled with iterative solvers, outperform traditional filters in many trajectory estimation problems. Their ability to integrate all available measurements in a batch optimization framework makes them particularly suitable for scenarios involving complex dynamics and sparse observations. However, their use in radar-based missile trajectory tracking is a relatively unexplored domain, motivating this study.

The present article aims to explore the smoothing-based solution as a viable alternative to KF approaches for the missile tracking and landing spot estimation problem. Its scope encompasses the mathematical formulation of the tracking problem within the framework of a factor graph, the simulation of radar-based missile observations, the evaluation of trajectory prediction and landing point estimation accuracy, and a comparative analysis with an Extended Kalman Filter (EKF), followed by a detailed analysis of the results.

Mathematical Formulation

A factor graph is a bipartite graph used to represent a problem as a structure comprising two types of nodes: hidden variables and factors. [3] Variable nodes represent the variables of interest to be estimated (e.g., state estimates of an object's trajectory), while factor nodes encode constraints (e.g., dynamic constraints) or measurements that relate to these variables.

Edges in a bipartite graph connect variable nodes exclusively to factor nodes, representing dependencies between specific variables and factors [3]. An edge from a variable node to a factor node indicates that the variable's value is involved in the factor's calculation or contributes to its outcome. This structure enables the decomposition of a complex global function into a product of simpler local functions, which facilitates efficient analysis and computation.

A factor graph representing the trajectory and landing point estimation problem is presented in Figure 1. Each factor models a specific aspect of the trajectory, such as the relationship between successive states (motion model) or the consistency with radar observations (measurement model). The factor graph includes:

1. **Variable nodes** representing the state of the missile at discrete time steps $\boldsymbol{x}_{0...N}$ which comprises the information about its position and velocity expressed in Cartesian coordinates.
2. **Factors**, which either represent a likelihood term derived from radar measurements $\boldsymbol{z}_{1...M}$ (red) or a transition model function $\boldsymbol{f}$ term encoding the dynamics of a given missile (blue).

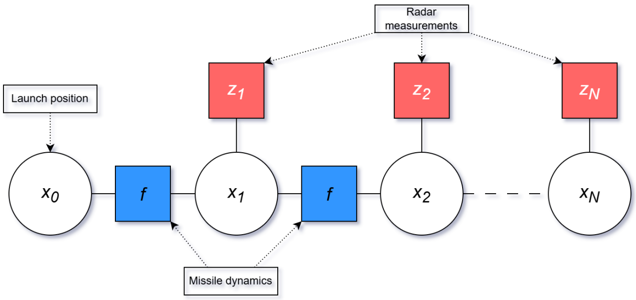

Fig. 1. The graph representing the described trajectory estimation problem

As a factor graph expresses the joint probability distribution of a set of variables as a product of factors, it can equivalently be formulated as an optimization problem. Specifically, the goal is to minimize the sum of residuals associated with all the factors:

For the trajectory estimation problem, this can be expressed as:

$$(1)\quad \hat{\chi} = \arg arg \min_{\chi} \left(\sum_{i=1}^{N} \frac{1}{2} \|f(x_{i-1}, u_{i-1}) - x_i\|_Q^2 +, \right.$$

where: $\boldsymbol{\chi}$ – set of possible states, N – number of dynamic constraints, $\boldsymbol{f}$ – dynamic transition vector function, – state vector at step, – external input vector at step, – process noise covariance matrix, M – number of measurements, $\boldsymbol{h}$ – measurement vector function, –z_k k-th measurement, R – measurement noise covariance matrix,

$\|\mathbf{e}\|_A^2$ notation denotes the norm of an error vector $\mathbf{e}$ weighted by its covariance matrix $\mathbf{A}$. Its usage is explained in equations (2) and (3), which represent cost functions derived from the factors:

$$(2)\quad \|f(x_{i-1}, u_{i-1}) - x_i\|_Q^2 = \left(f(x_{i-1}, u_{i-1}) - x_i\right)^T Q^{-1} \left(f(x_{i-1}, u_{i-1}) - x_i\right)$$

$$(3)\quad \|h(x_k) - z_k\|_R^2 = \left(h(x_k) - z_k\right)^T R^{-1} \left(h(x_k) - z_k\right)$$

Optimization involves finding the state configuration that minimizes the negative log-likelihood as a weighted least-squares algorithm.

In addition to formulating the factor graph as an optimization problem, another perspective is to represent it as a sparse matrix. Then, the estimation task can be described as iteratively solving the linear system:

$$(4)\quad J\Delta\hat{\chi} = \; - y$$

where: J – sparse matrix, which can be interpreted as equivalent adjacency matrix for a bipartite factor graph, $\Delta\hat{\chi}$ – is the update to the state variables, y – residual vector formed from stacked residuals.

The state variables are incrementally updated, obtaining the linearized solution, according to

$$(5)\quad \hat{\chi}_j = \hat{\chi}_{j-1} + \Delta\hat{\chi}_j$$

where: j – iteration index.

The sparse matrix approach takes advantage of the underlying structure of factor graphs, where the interactions between variables are typically sparse, meaning that each variable is influenced by only a small subset of factors.

This matrix J is constructed so that each hidden variable in the graph corresponds to a block of columns, each block of rows corresponds to a factor in the graph. Other matrix entries are zero unless there is an edge between the corresponding factor and hidden variable $\hat{x}_i$.

The matrix form is found by linearizing the error functions around a current estimate for a hidden variable:

$$(6)\quad \|f(x_{i-1}, u_{i-1}) - x_i\|_Q^2 = \|f(x_{i-1}, u_{i-1}) - \hat{x}_i + F_i\Delta x_i\|_Q^2$$

where: $\mathbf{F}_i$ – Jacobi matrix of the error function derived from dynamic constraints at step i.

$$(7)\quad \|h(x_k) - z_k\|_R^2 = \|h(\hat{x}_k) - z_k + H_k\Delta x_k\|_R^2$$

where: $\mathbf{H}_k$ – Jacobi matrix of the error function derived from dynamic constraints for the k-th measurement.

If inverses of covariance matrices $\mathbf{Q}$ and $\mathbf{R}$ are decomposed as follows into:

$$(8)\quad Q^{-1} = S_{Q^{-1}}^T S_{Q^{-1}}$$

$$(9)\quad R^{-1} = S_{R^{-1}}^T S_{R^{-1}}$$

then the elements of J corresponding to dynamic constraints at step i is can be denoted:

$$(10)\quad F_i' = S_{Q^{-1}} F_i$$

while the elements of J corresponding to k-th radar measurement would equal:

$$(11)\quad H_k' = S_{R^{-1}} H_k$$

where and are weighted derivative matrices.

An example of the matrix corresponding to one of performed simulations is shown in Figure 2.

The right-hand side of the normal equation (4), vector, is constructed using weighted residuals from dynamic constraints and measurements:

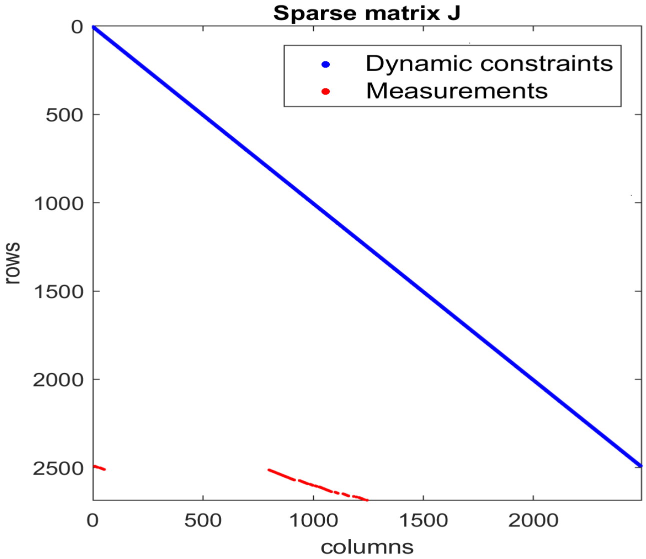

Fig. 2. The sparse matrix representing the described trajectory estimation problem.

(12)

$$y = \left[S_{Q^{-1}}\left(\left(x_0, u_1\right) - x_1\right) \ \cdots \ S_{Q^{-1}}\left(\left(x_{N-1}, u_N\right) - x_N\right) \ S_{R^{-1}}\left(h\left(x_1\right) - z\right.\right.$$

The normal equation (4) is typically solved using the sparsity leveraging methods, such as QR or Cholesky Decomposition [3].

QR decomposition factorizes the Jacobian matrix into an orthogonal matrix and an upper triangular matrix. This simplifies solving the normal equations to backward substitution, which is numerically stable and efficient.

Cholesky decomposition offers an alternative by factorizing a matrix into a lower triangular matrix. This method is computationally more efficient than QR decomposition [8]; however, it is only applicable to square matrices, and the Moore-Penrose pseudoinverse is often required as an intermediate step.

The proposed factor graph method was compared with an estimation tool built upon an EKF. This filter operates under the assumption that state transitions and measurements can be modelled as linear or locally linear processes, with Gaussian noise distributions, allowing for recursive updates to the state estimate and its associated uncertainty. Additionally, the EKF leverages the Markov process assumption, which states that the current state depends only on the immediately preceding state, simplifying the modeling of temporal dependencies.

The EKF is a well-established method, with its mathematical formulation extensively documented in the literature [6, 10–13]; therefore, the equations are not included here.

Simulation

Simulated missile trajectories were generated using a ballistic missile model described in [14] and implemented in MATLAB. This model incorporated key physical and environmental factors, including engine thrust, gravitational force, aerodynamic drag, and dynamic process noise. Radar observations were simulated as vectors containing range, azimuth, and elevation information. The measurement generation process also accounted for technical constraints, such as maximum radar range, elevation angle limitations, and the probability of target misdetection.

The key simulation parameters are provided in Table 1. The data related to missile range and maximum altitude reflect the variation resulting from different simulation runs with randomized flight parameters. For comparison purposes, data were collected from multiple realizations of each scenario, and the results were averaged to provide a comprehensive evaluation.

Table 1. Parameters used in the simulation

Parameter	Value	Units
Common parameters		
Standard deviation of measurement error – range	50	[m]
Standard deviation of measurement error – azimuth	0.3	[degrees]
Standard deviation of measurement error – elevation	0.5	[degrees]
Probability of detection	0.8	–
Maximum range	80 000	[m]
Maximum elevation	25	[degrees]
Sensor Update Rate	2	[Hz]
Radar coordinates	[0, 0, 0]	[m]
Missile launch site coordinates	[16000, 5000, 0]	[m]
Launch direction (azimuth angle)	105	[degrees]
Missile motion process noise intensity	70	[m^2/s^3]
Scenario 1 parameters		
Missile range	~38 000 – 40 000	[m]
Missile maximum Altitude	~19 500 – 21 000	[m]
Scenario 2 parameters		
Missile range	~64 000 – 71 000	[m]
Missile maximum altitude	~31 500 – 32 000	[m]
Scenario 3 parameters		
Missile range	~72 000 – 75 000	[m]
Missile maximum altitude	~33 000 – 35 000	[m]

An example of a simulated trajectory is presented in Figure 3.

Both the factor graph and the EKF trajectory estimation approaches were implemented, their performance was evaluated and compared in three operational scenarios. In the first scenario, the simulation was configured such that the entire missile trajectory was within radar range, allowing the radar to observe the missile's full path. The second scenario involved a partially visible trajectory, where the missile landing site was located at the edge of the radar's maximum range. The third and most challenging scenario featured minimal trajectory coverage, with only a small portion of the missile's flight path remaining within radar range and below the maximum elevation angle.

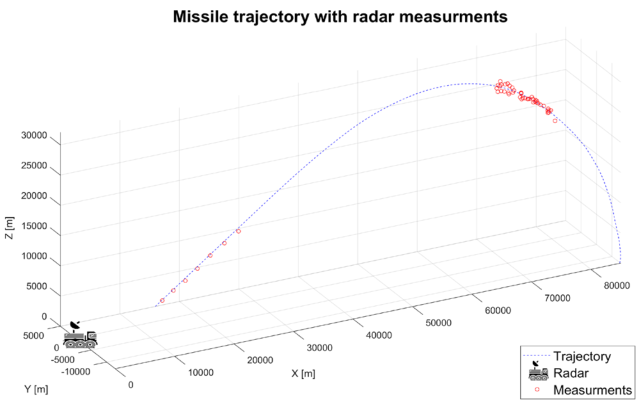

Fig. 3. Simulated trajectory and measurements

An example of the trajectory, generated for the most challenging scenario, and the estimation results obtained using the factor graph approach, along with successive trajectory optimization steps, is shown in Figure 4.

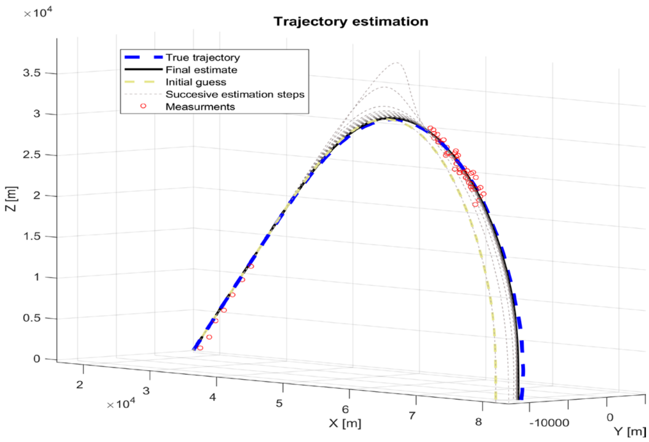

Fig. 4. Trajectory estimation using factor graph-based method.

Results

The accuracy of the factor graph and EKF approaches was compared using two criteria. First, the RMSE of the trajectory position error along the trajectory was calculated. Second, the mean error of the landing spot was evaluated. The results are presented in Table 2. The table also includes information on the height (above ground level, AGL) of the last measurement before impact.

Table 2. The results of accuracy comparison

Scenario 1 – last measurement AGL ~ 420 m		
Estimation tool	Trajectory RMSE [m]	Landing error [m]
Factor graph	10.08	145.28
EKF	17.91	157.56
Scenario 2 – last measurement AGL ~ 22 500 m		
Estimation tool	Trajectory RMSE [m]	Landing error [m]
Factor graph	19.23	763.94
EKF	30.48	1310.91
Scenario 3 – last measurement AGL ~ 32 000 m		
Estimation tool	Trajectory RMSE [m]	Landing error [m]
Factor graph	25.39	1193.62
EKF	65.02	5296.54

Conclusions

This study compares the factor graph-based trajectory estimation approach with the traditional EKF method for missile trajectory estimation using radar observations. The results clearly show that factor graphs consistently achieve lower RMSE values and smaller errors, particularly in the most challenging experimental setups with limited radar data. This highlights the effectiveness of factor graphs, which offer superior precision in estimating the missile's full trajectory.

While the EKF method is efficient for real-time applications due to its recursive nature, its reliance on the Markov assumption limits its ability to incorporate long-term historical information when making predictions. On the contrary, factor graphs employ a batch optimization approach that considers the entire trajectory at once, leading to more accurate point predictions and a more comprehensive estimate, even with limited data.

The findings underscore the potential of factor graphs for radar-based missile trajectory estimation, with possible applications extending to tracking cruise missiles with various motion models. Future work will focus on the real-time implementation of this method and its integration with hardware systems to enhance operational capability. The holistic, smoothing-based optimization approach employed by factor graphs offers a promising alternative to traditional methods, particularly for complex estimation problems in dynamic and constrained environments.

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