PRZEGLĄD ELEKTROTECHNICZNY

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On the determination of the general solutions for electromagnetic equations of rotary electric machines

O określaniu ogólnych rozwiązań równań elektromagnetycznych dla wirujących maszyn elektrycznych

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Abstract: This study presents an analytical method for determining transient solutions to electromagnetic equations applicable to electrical machines, based on Floquet theory for linear differential equations with periodic coefficients. Such equations are required to describe electrical machines operating at a constant rotational speed when the symmetry of the machine windings or magnetic circuit is disrupted. Floquet's theorem qualitatively defines transient solution; however, quantitative analyses using this theorem have been explored only to a very limited extent. Therefore, this study introduces an approach that enables the determination of transient solutions predicted using Floquet's theorem directly in the time domain, eliminating the need for numerical integration while retaining the advantages of qualitative algebraic analyses.

Keywords: periodic linear differential equations, Floquet's solution, time domain analysis, transients in electrical machines

Streszczenie: Artykuł prezentuje analityczną metodę określania rozwiązań nieustalonych dla równań elektromagnetycznych opisujących maszyny elektryczne bazując na teorii Floqueta dla liniowych równań różniczkowych o okresowo zmiennych współczynnikach. Równania takie opisują maszyny elektryczne pracujące przy stałej prędkości obrotowej gdy zakłócona jest symetria budowy uzwojeń lub obwodu magnetycznego. Teoria Floqueta określa jakościowo rozwiązania ogólne takich równań, jednak analiza ilościowa nie jest jednoznacznie określona. Artykuł przedstawia sposób algebraicznego obliczania rozwiązań nieustalonych w bezpośrednio w dziedzinie czasu, eliminując zastosowanie metod całkowania numerycznego, zachowując jednak cechy jakościowe analizy algebraicznej.

Słowa kluczowe: okresowe liniowe równania różniczkowe, rozwiązania Floqueta, analiza w dziedzinie czasu, stany nieustalone w maszynach elektrycznych

Introduction

Rotary motion electromechanical systems represent a broad class of equipment utilised in engineering applications. Foremost among these are electromechanical energy converters – particularly electric machines – that convert energy bidirectionally between electrical and mechanical forms. The equations governing rotary electric machines are typically expressed in a general form as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{E_{\mathrm{co}}(\varphi,\mathbf{i})}{\partial \mathbf{i}} + \mathbf{R}\cdot\mathbf{i} = \mathbf{u}(t) \tag{1.1a}$$

$$\mathrm{J}\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} + \mathrm{D}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{E_{\mathrm{co}}(\varphi,\mathbf{i})}{\partial\varphi} + t_{\mathrm{m}}(t,\varphi), \tag{1.1b}$$

where:
$E_{\mathrm{co}}(\varphi, \mathbf{i})$ – is the co-energy function,
$\mathbf{i}(t) = [i_1(t)\ \ i_2(t) \cdots i_N(t)]^T$ – is the vector of currents,
$\mathbf{u}(t) = [u_1(t)\ \ u_2(t) \cdots u_{\mathrm{N}}(t)]^T$ – is the vector of voltages,
$\mathbf{R}$ – is the matrix of winding resistances,
φ – is the rotary angle,
J – is the moment of inertia of the rotor,
D – is the damping factor of motion, and
$t_{\mathrm{m}}(t,\varphi)$ is the external mechanical torque.

Electromagnetic processes in electrical machines are frequently analysed under the assumption of constant rotor speed $\varphi = \Omega_{\mathrm{m}} \cdot t + \varphi_0$, meaning that the solution to the mechanical equation is assumed to be known and can be omitted. Under this assumption, the electromagnetic equations take the following form.

$$\mathbf{\Psi}'(\varphi,\mathbf{i}) + \mathbf{R}\cdot\mathbf{i} = \mathbf{u}(t). \tag{1.2}$$

$\mathbf{\Psi} = [\psi_1(\varphi,\mathbf{i})\ \ \psi_2(\varphi,\mathbf{i}) \cdots \psi_{\mathrm{N}}(\varphi,\mathbf{i})]^T$ is the vector of the flux linkages, which is often linearized to the relation

$$\Psi(\varphi, \mathbf{i}) = \mathbf{L}(\varphi) \cdot \mathbf{i}, \tag{1.3}$$

where $\mathbf{L}(\varphi)$ is the matrix of the winding inductances, satisfying the condition

$$\mathbf{L}(\varphi) = \mathbf{L}(t) = \mathbf{L}(t + \mathrm{T_m}) \text{ at } \mathrm{T_m} - 2\pi/\Omega_\mathrm{m}.$$

Under such conditions, electromagnetic equations governing electrical machines reduce to linear differential equations with periodic coefficients of the form

$$\boldsymbol{\Psi}' = \mathbf{A}(t) \cdot \boldsymbol{\Psi} + \mathbf{u}(t), \tag{1.4}$$

where $\mathbf{A}(t) = -\mathbf{R} \cdot (\mathbf{L}(t))^{-1} = \mathbf{A}\,(t + \mathrm{T_m})$.

The inductance matrix $\mathbf{L}(\varphi)$ of electrical machines always has a positive determinant for physical reasons. Therefore, the matrix $\mathbf{A}(t)$ is defined and unique. Considering that Equation (1.4) is linear, its solution can be written in the following form

$$\boldsymbol{\Psi}(t) = \mathbf{K}(t,0)\,((\boldsymbol{\Psi}(t) - \boldsymbol{\Psi}_\mathrm{p}(0)) + \boldsymbol{\Psi}_\mathrm{p}(t)) \tag{1.5}$$

where $\mathbf{K}(t,0)$ represents the Cauchy matrix

$$\mathbf{K}(t,0) = \mathbf{X}(t) \cdot \mathbf{X}^{-1}(0).$$

The matrix $\mathbf{X}(t)$ comprises N independent solutions of the homogenous equation, i.e., when the forcing vector $\mathbf{u}(t)$ is omitted, and $\boldsymbol{\Psi}_\mathrm{p}(t)$ is the vector of a particular solution corresponding to the forced voltage vector $\mathbf{u}(t)$. Moreover, $\boldsymbol{\Psi}_\mathrm{p}(t)$ represents the steady-state solution of equation (1.4), obtained after the transient components have vanished.

Particular solutions are highly critical from an engineering standpoint because they determine the steady-state properties of electrical machines subjected to periodic or quasi-periodic voltages. Notably, several methods exist to determine these solutions [1]. However, most of these methods are based on the prediction of specific features of steady-state behaviours and the application of harmonic balance approaches to find the solutions. By contrast, methods for finding transient solutions using the Cauchy matrix are less popular for several reasons. Electromagnetic equations are typically solved along with mechanical equations, thereby forming nonlinear systems (1.1) requiring numerical integration. Additionally, the magnetic circuit of an electrical machine is typically nonlinear, necessitating the linearization of relationships between flux linkages and currents to be compatible with the inductance matrix expressed in Equation (1.4). For symmetrically designed machines, which are of primary interest from an engineering standpoint, Equation (1.4) can be reduced to linear differential equations with constant coefficients. The class of orthogonal or unitary linear transformations, which is well-known in electrical engineering [1] enables the study of transients (for that class of equations) through fundamentally algebraic methods, thereby providing quantitative insight into the constituent electromagnetic phenomena. However, when symmetry is lost, the equations in the form of Equation (1.4) must be solved. Floquet's theorem [2] [3] can be used to determine the general solution qualitatively, but quantitative analyses using this theorem are not well developed as for linear equations with constant coefficients. Therefore, this study presents an approach to determine the Cauchy matrix, predicted using Floquet's theorem, directly in the time domain, preventing numerical integration while simultaneously retaining the benefits of qualitative algebraic analyses.

Problem formulation

The first-order linear differential equation under consideration is

$$\mathbf{x}'(t) = \mathbf{A}(t) \cdot \mathbf{x}(t) + \mathbf{f}(t) \tag{2.1}$$

where:
$\mathbf{x}(t) = [x_1(t)\ \ x_2(t)\ \cdots\ x_\mathrm{N}(t)]^T$ – is the vector of solutions,
$\mathbf{x}'(t) = [x_1'(t)\ \ x_2'(t)\ \cdots\ x_\mathrm{N}'(t)]^T$ – is the vector of first derivatives,
$\mathbf{f}(t) = [f_1(t)\ \ f_2(t)\ \cdots f_\mathrm{N}(t)]^T$ – is a forcing function vector,
$\mathbf{A}(t) = \mathrm{A}(t + \mathrm{T})$ – is a square periodic matrix.

Its solution can be written in the form

$$\mathbf{x}(t) = \mathbf{K}(t,0)\big((\mathbf{x}(t) - \mathbf{x}_\mathrm{p}(0)) + \mathbf{x}_\mathrm{p}(t)\big), \tag{2.2}$$

where $\mathbf{K}(t,0)$ represents the Cauchy matrix, and $\mathbf{x}_\mathrm{p}(t)$ is the vector of a particular solution corresponding to the forcing function vector $\mathbf{f}(t)$. The Cauchy matrix is determined by the fundamental matrix $\mathbf{X}(t)$, which comprises N independent solutions of the homogenous equation

$$\mathbf{x}'(t) = \mathbf{A}(t) \cdot \mathbf{x}(t) \tag{2.3}$$

and assumes the form

$$\mathbf{K}(t,0) = \mathbf{X}(t) \cdot \mathbf{X}^{-1}(0).$$

As has been established via Floquet's theorem, the homogenous equation with a periodic matrix $\mathbf{A}(t)$ can be reduced using the linear transformation [4] [5] [6] and [7]

$$\mathbf{x}(t) = \mathbf{S}(t) \cdot \mathbf{y}(t) \tag{2.4}$$

to the linear equation with a constant matrix **B** as

$$\mathbf{y}'(t) = \mathbf{B} \cdot \mathbf{y}(t). \tag{2.5}$$

Subsequently, the fundamental matrix $\mathbf{X}(t)$ can be expressed in the form of

$$\mathbf{X}(t) = \mathbf{S}(t) \cdot e^{\mathbf{B} \cdot t} \tag{2.6}$$

where $e^{\mathbf{B} \cdot t}$ is a matrix function of $\mathbf{B} \cdot t$, and the transformation matrix is periodic, i.e., $\mathbf{S}(t) = \mathbf{S}(t + \mathrm{T})$. The matrices $\mathbf{A}(t)$, $\mathbf{S}(t)$, and $\mathbf{S}^{-1}(t)$ fulfil the relationship

$$\mathbf{B} = \mathbf{S}^{-1}(t) \cdot \mathbf{A}(t) \cdot \mathbf{S}(t) - \mathbf{S}^{-1}(t) \cdot \mathbf{S}'(t). \tag{2.7}$$

Finally, the Cauchy matrix given in equation (2.2) can be expressed as

$$\mathbf{K}(t, 0) = \mathbf{S}(t) \cdot e^{\mathbf{B} \cdot t} \cdot \mathbf{S}^{-1}(0). \tag{2.8}$$

Notably, this study focuses on determining the matrices $\mathbf{S}(t)$ and **B** directly from the matrix $\mathbf{A}(t)$ in the time domain.

Assuming that the time-dependent matrices $\mathbf{A}(t)$, $\mathbf{S}(t)$, and $\mathbf{S}^{-1}(t)$ can be expanded into uniformly convergent Fourier series of the forms

$$\mathbf{A}(t) = \mathbf{A}(t + \mathrm{T})) = \sum_{r=-\infty}^{\infty} \mathbf{A}_r \cdot e^{\mathrm{j} \cdot r \cdot \Omega \cdot t},$$

$$\mathbf{S}(t) = \mathbf{S}(t + \mathrm{T}) = \sum_{r=-\infty}^{\infty} \mathbf{S}_r \cdot e^{\mathrm{j} \cdot r \cdot \Omega \cdot t},$$

$$\mathbf{S}^{-1}(t) = \mathbf{S}^{-1}(t + \mathrm{T}) = \sum_{r=-\infty}^{\infty} \bar{\mathbf{S}}_r \cdot e^{\mathrm{j} \cdot r \cdot \Omega \cdot t}$$

Equation (2.7) has been reinterpreted as relationships between the Fourier coefficients of those series [8], i.e., in the frequency domain, as (2.9), where $\Omega = 2\pi/\mathrm{T}$, and **E** is the (N×N) unit matrix. Equality (2.9) holds for infinite-dimen sional matrices; however, these matrices can be limited to finite square dimensions. Matrices **B** and $\mathbf{S}(t)$ can be determined by solving the eigenvalue problem for the matrix on the left-hand side, which is constructed on the basis of the matrix $\mathbf{A}(t)$, as reported in [8] [1]. This procedure effectively extends the conventional method for finding the general solution of differential equations with constant coefficients. This approach is conducted in the frequency domain, for which the Fourier coefficients of the matrix $\mathbf{A}(t)$ must first be determined from its time-domain form. The resulting matrix $\mathbf{S}(t)$ is then also determined in the frequency domain.

This study proposes a modified approach for determining the solution only in the time domain, which eliminates the need to calculate the Fourier coefficients. For such a time-domain-based approach, the Fourier series expressed in (2.7) should be reduced to a finite number of terms, up to $r = \mathrm{R}$, exemplary

$$\mathbf{A}(t) = \sum_{r=-\mathrm{R}}^{\mathrm{R}} \mathbf{A}_r \cdot e^{\mathrm{j} \cdot r \cdot \Omega \cdot t}. \tag{2.10}$$

These limited Fourier series possess 2R+1 coefficients and establish unique relationships with the values of the Fourier series in 2R+1 time instants uniformly distributed over the period [10]. For example, at $t_r = r \cdot \Delta\mathrm{T}$ for $-\mathrm{R} \le r \le \mathrm{R}$ and $\Delta\mathrm{T} = \mathrm{T}/(2\mathrm{R} + 1)$, the matrix $\mathbf{A}(t)$ is of the form

$$\mathbf{A}_t = \mathbf{C} \cdot \mathbf{A}_F \tag{2.11}$$

where:

$\mathbf{A}_t$ – is a vector formed from the matrices $\mathbf{A}(t_r)$, and

$\mathbf{A}_t = [\mathbf{A}(t_R) \cdots \mathbf{A}(t_{-1})\ \ \mathbf{A}(t_0)\ \ \mathbf{A}(t_1) \cdots \mathbf{A}(t_{-R})]^T$

$\mathbf{A}_F$ – is a vector formed from the matrices $\mathbf{A}_r$, and

$\mathbf{A}_t = [\mathbf{A}_R \cdots \mathbf{A}_1\ \ \mathbf{A}_0\ \ \mathbf{A}_{-1} \cdots \mathbf{A}_{-R}]^T$; and

C – is a unitary matrix of the form

$$\begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \mathbf{A}_0 - \mathrm{j}\Omega\mathbf{E} & \mathbf{A}_1 & \mathbf{A}_2 & \cdots \\ \cdots & \mathbf{A}_{-1} & \mathbf{A}_0 & \mathbf{A}_1 & \cdots \\ \cdots & \mathbf{A}_{-2} & \mathbf{A}_{-1} & \mathbf{A}_0 + \mathrm{j}\Omega\mathbf{E} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \mathbf{S}_0 & \mathbf{S}_1 & \mathbf{S}_2 & \cdots \\ \cdots & \mathbf{S}_{-1} & \mathbf{S}_0 & \mathbf{S}_1 & \cdots \\ \cdots & \mathbf{S}_{-2} & \mathbf{S}_{-1} & \mathbf{S}_0 & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} \ddots & & & \\ & \mathbf{B} - \mathrm{j}\Omega\mathbf{E} & & \\ & & \mathbf{B} & \\ & & & \mathbf{B} + \mathrm{j}\Omega\mathbf{E} \end{bmatrix} \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \bar{\mathbf{S}}_0 & \bar{\mathbf{S}}_1 & \bar{\mathbf{S}}_2 & \cdots \\ \cdots & \bar{\mathbf{S}}_{-1} & \bar{\mathbf{S}}_0 & \bar{\mathbf{S}}_1 & \cdots \\ \cdots & \bar{\mathbf{S}}_{-2} & \bar{\mathbf{S}}_{-1} & \bar{\mathbf{S}}_0 & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} \tag{2.9}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{c}^{\mathrm{R}^2} & \cdots & \mathbf{c}^{\mathrm{R}} & \mathbf{c}^0 & \mathbf{c}^{-\mathrm{R}} & \cdots & \mathbf{c}^{-\mathrm{R}^2} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ \mathbf{c}^{\mathrm{R}} & \cdots & \mathbf{c}^1 & \mathbf{c}^0 & \mathbf{c}^{-1} & \cdots & \mathbf{c}^{-\mathrm{R}} \\ \mathbf{c}^0 & \cdots & \mathbf{c}^0 & \mathbf{c}^0 & \mathbf{c}^0 & \cdots & \mathbf{c}^0 \\ \mathbf{c}^{-\mathrm{R}} & \cdots & \mathbf{c}^{-1} & \mathbf{c}^0 & \mathbf{c}^1 & \cdots & \mathbf{b}^{\mathrm{R}} \\ \vdots & \iddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}^{-\mathrm{R}^2} & \cdots & \mathbf{c}^{-\mathrm{R}} & \mathbf{c}^0 & \mathbf{c}^{\mathrm{R}} & \cdots & \mathbf{c}^{\mathrm{R}^2} \end{bmatrix}$$

$$c = e^{j\cdot\frac{2\pi}{2R+1}},\ \mathbf{c}^r = c^r \cdot \mathbf{E},\ C^{-1} = \frac{1}{(2R+1)}\ (\overset{*}{\mathbf{C}^T}).$$

The same type of relationships apply for the series $\mathbf{S}(t)$ and $\mathbf{S}^{-1}(t)$.

Equation (2.7) can be rewritten as

(2.12) $$\mathbf{S}'(t) = \mathbf{A}(t) \cdot \mathbf{S}(t) - \mathbf{S}(t) \cdot \mathbf{B},$$

which provided the basis to determine a relationship equivalent to equation (2.9), except in the time domain. It represents a set of differential equations of the form

(2.13) $$\mathbf{S}'_n(t) = \mathbf{A}(t) \cdot \mathbf{S}_n(t) - \mathbf{S}_n(t) \cdot \mathbf{B},$$

$\mathbf{S}_n(t) = [\mathbf{S}_{n,1}(t) \cdots \mathbf{S}_{n,\mathrm{N}}(t)]^T$ – are columns of the matrix $\mathbf{S}(t)$, and

(2.14) $$\mathbf{S}(t) = [\mathbf{S}_1\ (t) \cdots \mathbf{S}_{\mathrm{N}}(t)].$$

Matrix **B** can be predicted in its simplest Jordan form as

(2.15) $$\mathbf{B} = \mathrm{diag}[\gamma_1 \cdots \gamma_2 \cdots \gamma_{\mathrm{N}}],$$

and equation (2.12) can be rewritten as

(2.16) $$\mathbf{S}'_n(t) = (\mathbf{A}(t) - \lambda_n \cdot \mathbf{E}) \cdot \mathbf{S}_n(t),$$

$$\text{for } n = \{1, 2, \ldots, \mathrm{N}\}.$$

For an individual instant of time $\mathrm{tr} = r \cdot \Delta\mathrm{T}$, equation (2.13) assumes the form

$$\mathbf{S}'_n(t_r) = (\mathbf{A}(t_r) - \lambda_n \cdot \mathbf{E}) \cdot \mathbf{S}_n(t_r),$$

$$\text{for } n = \{1, 2, \ldots, \mathrm{N}\} \text{ and } -\mathrm{R} \le r \le \mathrm{R}.$$

The vectors of derivatives $\mathbf{S}'_n(t_r)$ can be approximated using the discrete differential operator [10], which relates the values of the first derivative to the values of the function at a chosen set of time instants, i.e.,

(2.18) $$\mathbf{S}'_{n,t} = \mathbf{D} \cdot \mathbf{S}_{n,t},$$

where

$\mathbf{S}'_{n,t} = [\mathbf{S}'_n\,(t_{\mathrm{R}}) \cdots \mathbf{S}'_n\,(t_{-1})\ \ \mathbf{S}'_n\,(t_0)\ \ \mathbf{S}'_n\,(t_1) \cdots \mathbf{S}'_n\,(t_{-\mathrm{R}})]^T$,

$\mathbf{S}_{n,t} = [\mathbf{S}_n\,(t_{\mathrm{R}}) \cdots \mathbf{S}_n\,(t_{-1})\ \ \mathbf{S}_n\,(t_0)\ \ \mathbf{S}_n\,(t_1) \cdots \mathbf{S}_n\,(t_{-\mathrm{R}})]^T$,

D – is a singular matrix with dimensions (2R+1)× (2R+1) of the form

$$\mathbf{D} = \begin{bmatrix} \mathbf{0} & -\mathbf{d}_1 & \cdots & -\mathbf{d}_{\mathrm{R}} & \mathbf{d}_{\mathrm{R}} & \cdots & \mathbf{d}_1 \\ \mathbf{d}_1 & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{0} & -\mathbf{d}_1 & \vdots & \ddots & \mathbf{d}_{\mathrm{R}} \\ \mathbf{d}_{\mathrm{R}} & \vdots & \mathbf{d}_1 & \mathbf{0} & -\mathbf{d}_1 & \vdots & -\mathbf{d}_{\mathrm{R}} \\ -\mathbf{d}_{\mathrm{R}} & \ddots & \vdots & \mathbf{d}_1 & \mathbf{0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & -\mathbf{d}_1 \\ -\mathbf{d}_1 & \cdots & -\mathbf{d}_{\mathrm{R}} & \mathbf{d}_{\mathrm{R}} & \cdots & \mathbf{d}_1 & \mathbf{0} \end{bmatrix}$$

$\mathbf{d}_r$ is the set of (N×N) diagonal matrices with element d_r on the main diagonal

$$d_r = \Omega \cdot \frac{2}{2\mathrm{R}+1} \sum_{k=1}^{\mathrm{R}} k \cdot \sin\left(k \cdot r \cdot \frac{2\pi}{2\mathrm{R}+1}\right).$$

Equation (2.17) is then reduced to the form

(2.19) $$(\mathbf{A}_t - \mathrm{D} - \lambda_n \cdot \mathbf{E}) \cdot \mathbf{S}_{n,t} = 0,$$

$$\text{for } \{1, 2, \ldots, \mathrm{N}\}.$$

but now

$$\mathbf{A}_t = \mathrm{diag}[\mathbf{A}(t_{\mathrm{R}}) \cdots \mathbf{A}(t_{-1})\ \ \mathbf{A}(t_0)\ \ \mathbf{A}(t_1) \cdots \mathbf{A}(t_{-\mathrm{R}})].$$

All the above equations are singular and share the same singular matrix $(\mathbf{A}_t - \mathbf{D} - \lambda \cdot \mathbf{E})$. This means that the problem reduces to solving the eigenvalue problem for the matrix $(\mathbf{A}_t - \mathbf{D})$. Here, the eigenvalues must satisfy the equation (2.20). The matrix $(\mathbf{A}_t - \mathbf{D})$ can typically have N×(2R+1) eigenvalues. The λ_n values should be determined through numerical tests, based on the aforementioned set of eigenvalues.

The corresponding eigenvectors $\mathbf{S}_n(t)$, at the set of points $t_r = r \cdot \Delta\mathrm{T}$ for $-\mathrm{R} \le r \le \mathrm{R}$ and $\Delta\mathrm{T} = \mathrm{T}/(2\mathrm{R}+1)$ can be calculated from the system of singular equations (2.21).

(2.20)
$$\det\begin{bmatrix} \mathbf{A}(t_R)-\lambda\mathbf{E} & \mathbf{d}_1 & \cdots & \mathbf{d}_R & -\mathbf{d}_R & \cdots & -\mathbf{d}_1 \\ -\mathbf{d}_1 & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{A}(t_1)-\lambda\mathbf{E} & \mathbf{d}_1 & \vdots & \ddots & -\mathbf{d}_R \\ -\mathbf{d}_R & \vdots & -\mathbf{d}_1 & \mathbf{A}(t_0)-\lambda\mathbf{E} & \mathbf{d}_1 & \vdots & \mathbf{d}_R \\ \mathbf{d}_R & \ddots & \vdots & -\mathbf{d}_1 & \mathbf{A}(t_{-1})-\lambda\mathbf{E} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \mathbf{d}_1 \\ \mathbf{d}_1 & \cdots & \mathbf{d}_R & -\mathbf{d}_R & \cdots & -\mathbf{d}_1 & \mathbf{A}(t_{-R})-\lambda\mathbf{E} \end{bmatrix} = 0\,, \quad \text{for} \quad n=\{1,2,\ldots,\mathrm{N}\}.$$

(2.21)
$$\begin{bmatrix} \mathbf{A}(t_R)-\lambda_n\cdot\mathbf{E} & \mathbf{d}_1 & \cdots & \mathbf{d}_R & -\mathbf{d}_R & \cdots & -\mathbf{d}_1 \\ -\mathbf{d}_1 & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{A}(t_1)-\lambda_n\cdot\mathbf{E} & \mathbf{d}_1 & \vdots & \ddots & -\mathbf{d}_R \\ -\mathbf{d}_R & \vdots & -\mathbf{d}_1 & \mathbf{A}(t_0)-\lambda_n\cdot\mathbf{E} & \mathbf{d}_1 & \vdots & \mathbf{d}_R \\ \mathbf{d}_R & \ddots & \vdots & -\mathbf{d}_1 & \mathbf{A}(t_{-1})-\lambda_n\cdot\mathbf{E} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \mathbf{d}_1 \\ \mathbf{d}_1 & \cdots & \mathbf{d}_R & -\mathbf{d}_R & \cdots & -\mathbf{d}_1 & \mathbf{A}(t_{-R})-\lambda_n\cdot\mathbf{E} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{S}_{n,R} \\ \vdots \\ \mathbf{S}_{n,1} \\ \mathbf{S}_{n,0} \\ \mathbf{S}_{n,-1} \\ \vdots \\ \mathbf{S}_{n,-R} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$
$$\text{for} \ \ n=\{1,2,\ldots,\mathrm{N}\}.$$

The unique solution thereof requires the assumption of the value of one of the unknown vectors. The eigenvectors can also be determined via numerical integration of differential equation (2.16) for individual eigenvalues, assuming $\mathbf{S}_n(0)$, as

(2.22) $$\mathbf{S}'_n(t) = (\mathbf{A}(t) - \lambda_n \cdot \mathbf{E}) \cdot \mathbf{S}_n(t),$$

$$\text{for } n = \{1, 2, \ldots, \mathrm{N}\}$$

Numerical tests

A. Description of solution

The relationships presented earlier were tested to determine the general solution of the equation for the simplest reluctance electromechanical converter exhibiting rotary motion at a constant angular speed, namely,

(3.1) $$u(t) = (L(\varphi) \cdot i)' + \mathrm{R_s} \cdot i,$$

where $L(\varphi) = \mathrm{L}_0 + \mathrm{L}_2 \cdot \cos 2\varphi$ and $\varphi = \Omega_m \cdot t + \varphi_0$. When the flux linkage $\psi = \mathrm{L}(t) \cdot i$ is used as a variable, the homogeneous form of this equation assumes the form

(3.2) $$\psi' + (\mathrm{R_s}/L(t)) \cdot \psi = 0$$

where $L(t) = L(t + \mathrm{T_m})$ and $\mathrm{T_m} = 2\pi/\Omega_m$, which constitutes the linear differential equation with periodic coefficients in the normal form

(3.3) $$\psi' = a(t) \cdot \psi\,,$$

$$a(t) = -\mathrm{R_s}/(\mathrm{L}_0 + \mathrm{L}_2 \cdot \cos 2(\Omega_m \cdot t + \varphi_0)).$$

Its general solution, according to equations (2.2) and (2.6), is of the form

(3.4) $$\psi = k(t,0) \cdot \psi_0\,,$$

where $k(t,0) = (s(t)/s(0)) \cdot e^{\lambda \cdot t}$ and $\psi_0 = \psi(0)$. The function $s(t)$ and the value of λ satisfy differential equation (2.16), which reduces to

(3.5) $$s'(t) = (a(t) - \lambda) \cdot s(t)).$$

On approximating the derivative on the left-hand side by the function values $s(t_r)$ for a set of 2R+1 points $t_r = r \cdot \Delta\mathrm{T_m}$ uniformly located along the period $\mathrm{T_m}$ for $-\mathrm{R} \le r \le \mathrm{R}$, where $\Delta\mathrm{T^m} = \mathrm{T_m}/(2\mathrm{R} + 1)$, and using the discrete differential operator, the aforementioned differential equation reduces to a system of algebraic equations as

(3.6) $$(\mathbf{a}_t - \mathbf{D} - \lambda_n \cdot \mathbf{E}) \cdot \mathbf{s}_t = 0,$$

where: $\mathbf{a}_t = \mathrm{diag}[a(t_R) \cdots a(t_{-1})\ \ a(t_0)\ \ a(t_1) \cdots a(t_{-R})]^T$.
and $\mathbf{s}_t = [s(t_R) \cdots s(t_{-1})\ \ s(t_0)\ \ s(t_1) \cdots s(t_{-R})]^T$.

To determine λ and $s(t_r)$, the eigenvalue problem for the matrix $(\mathbf{a}_t - \mathbf{D})$ must be solved. The eigenvalue λ must satisfy the equality (3.7)

$$\det\begin{bmatrix} a(t_R)-\lambda & d_1 & \cdots & d_R & -d_R & \cdots & -d_1 \\ -d_1 & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & a(t_1)-\lambda & d_1 & \vdots & \ddots & -d_R \\ -d_R & \vdots & -d_1 & a(t_0)-\lambda & d_1 & \vdots & d_R \\ d_R & \ddots & \vdots & -d_1 & a(t_{-1})-\lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & d_1 \\ d_1 & \cdots & d_R & -d_R & \cdots & -d_1 & a(t_{-R})-\lambda \end{bmatrix} = 0 \tag{3.7}$$

where $d_r = \Omega_m \cdot \frac{2}{2R+1} \sum_{k=1}^{R} k \cdot \sin\left(k \cdot r \cdot \frac{2\pi}{2R+1}\right)$.

However, as the algebraic system in (3.6) only approximates the differential equation (3.5), it is expected that the resulting matrix ($\mathbf{a}_t$ – $\mathbf{D}$) can have 2R + 1 different eigenvalues λ_r. The appropriate eigenvalue must be chosen through numerical testing. Most likely, it will correspond to the eigenvalue λ_0.

Vector $\mathbf{s}_t$ contains the values of the function $s(t)$ at the chosen set of time points $\{t_r\}$, which can be determined from the set of singular algebraic equation (3.8) for the selected eigenvalue, assuming an arbitrary value for one of the values of $s(t_r)$.

$$\begin{bmatrix} a(t_R)-\lambda_0 & d_1 & \cdots & d_R & -d_R & \cdots & -d_1 \\ -d_1 & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & a(t_1)-\lambda_0 & d_1 & \vdots & \ddots & -d_R \\ -d_R & \vdots & -d_1 & a(t_0)-\lambda_0 & d_1 & \vdots & d_R \\ d_R & \ddots & \vdots & -d_1 & a(t_{-1})-\lambda_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & d_1 \\ d_1 & \cdots & d_R & -d_R & \cdots & -d_1 & a(t_{-R})-\lambda_0 \end{bmatrix} \cdot \begin{bmatrix} s(t_R) \\ \vdots \\ s(t_1) \\ s(t_0) \\ s(t_{-1}) \\ \vdots \\ s(t_{-R}) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.8}$$

Alternatively, function $s(0)$ can be determined through numerical integration of Equation (3.5) as

$$s'(t) = (a(t) - \lambda_0) \cdot s(t)) \tag{3.9}$$

at $s(0) = s_0$ for $0 \le t \le T_m$

B. Results of calculations

The calculations have been performed for the converter considering the following (more or less realistic), per unit parameters: R_s = 0,1, L_0 = 1,0, L_2 = 0,25 and T_m = 1.0.

The appropriate value of λ is determined by solving (3.7) for the series of values R, which define the highest harmonic accounting for the function $a(t)$. Only even values are considered because only even harmonics are expected in the function $a(t)$. For successive R values, 2R+1 different eigenvalues λ are obtained, denoted as $\{\lambda_{-R},\ldots, \lambda_{-1}, \lambda_0, \lambda_1,\ldots, \lambda_R\}$. The eigenvalues have been ordered to satisfy the condition $\lambda_r = \lambda_{-r}$ owing to the symmetry of matrix (3.7). Then, the unique eigenvalue λ_0 can be determined. Subsequently, the pairs $\lambda_r = \lambda_{-r}$ are ordered based on the order of differences $\Delta_r = \lambda_r - \lambda_0$. The calculation results show that all the λ_r values are very close to one another; moreover, for a sufficiently large R, the eigenvalues $\lambda_{\pm1}$, $\lambda_{\pm2}$ take the value λ_0 with the assumed accuracy. Table 3.1 presents the eigenvalues $\{\lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2\}$ for R = {2, 4, 6, 8, 10}.

Table 3.2 lists the differences $\Delta\lambda_0 = |\lambda_0 - \lambda_R|$, which confirm that all the eigenvalues are very similar.

Finally, the value $\Delta\lambda_0$ = –0,10328 has been accepted for further calculations, on the basis of which the function $s(t)$ is calculated, using the equation set (3.8) as well as the numerical integration of (3.9), considering $s(0)$ = 1.0 for both cases. However, special care is required when solving (3.8). This equation set is singular; therefore, even if $s(0)$ = 1.0 is assumed, the determinants of the new equations are very close to zero because of the minuscule differences between neighbouring eigenvalues. In this case, an iterative normalisation technique for the scaling factor proves sufficient to obtain the solution. The function $s(t)$ is shown in Fig. 3.1, wherein the solid line represents the waveform constructed from the Fourier series calculated from the values $s(t_r)$, in turn obtained by solving (3.8). Meanwhile, the discrete points represent the values obtained via numerical integration of Equation (3.9).

Figure 3.2 presents the current waveform $i(t)$ following a short circuit of the converter coil, starting from the initial value $i(0)$ = 1.0, calculated using $s(t)$ and λ_0. Fig. 3.3 illustrates the selected points from the transient simulation along this waveform.

The waveforms determined using Floquet's formula closely match the simulation results, exhibiting a near-complete overlap.

Table 3.1 Eigenvalues $\lambda_0, \lambda_{\pm1}, \lambda_{\pm2}$,

	λ_{-2}	λ_{-1}	λ_0	λ_1	λ_2
R = 2	–0,10327272610351	–0,10327272054940	–0,10327274305779	–0,10327272054940	–0,10327272610351
R = 4	–0,10327955747716	–0,10327955739226	–0,10327955429864	–0,10327955739226	–0,10327955747716
R = 6	–0,10327955595425	–0,10327955595314	–0,10327955590067	–0,10327955595314	–0,10327955595425
R = 8	–0,10327955589970	–0,10327955589969	–0,10327955589889	–0,10327955589969	–0,10327955589970
R = 10	–0,10327955589887	–0,10327955589887	–0,10327955589886	–0,10327955589887	–0,10327955589887

Table 3.2 Differences $\Delta\lambda_0 = |\lambda_0 - \lambda_R|$

R = 2	R = 4	R = 6	R = 8	R = 10
$\Delta\lambda = \lvert\lambda_0 - \lambda_2\rvert$	$\Delta\lambda = \lvert\lambda_0 - \lambda_4\rvert$	$\Delta\lambda = \lvert\lambda_0 - \lambda_6\rvert$	$\Delta\lambda = \lvert\lambda_0 - \lambda_8\rvert$	$\Delta\lambda = \lvert\lambda_0 - \lambda_{10}\rvert$
$1{,}695427 \cdot 10^{-8}$	$3{,}536112 \cdot 10^{-9}$	$2{,}680657 \cdot 10^{-9}$	$2{,}420652 \cdot 10^{-9}$	$2{,}284764 \cdot 10^{-9}$

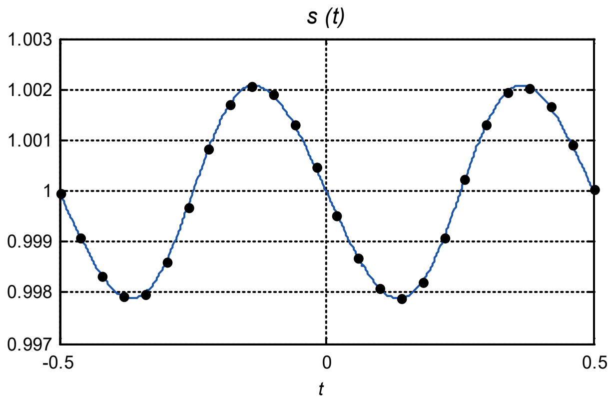

Fig. 3.1. Representation of function $s(t)$

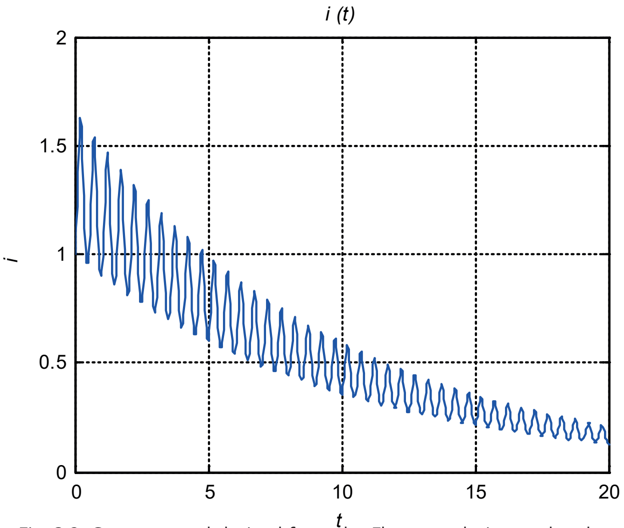

Fig. 3.2. Current trend derived from the Floquet solution under short circuit

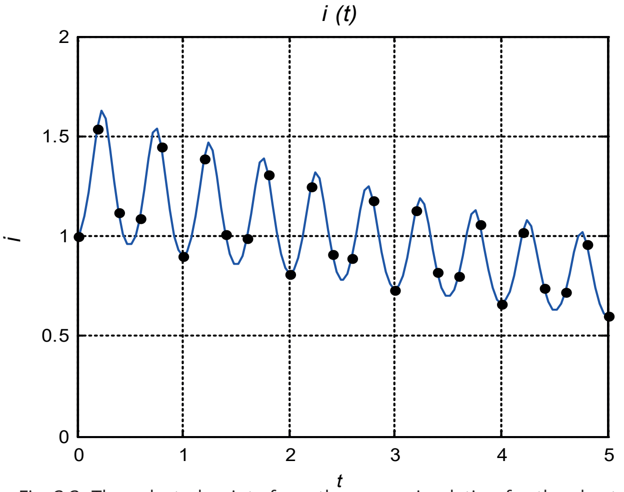

Fig. 3.3. The selected points from the curve simulation for the short-circuit scenario

Conclusions

This study proposes a technique to determine the general solution to periodic linear differential equations as predicted via Floquet's theorem. This technique qualitatively and quantitatively determines the features of transient processes based on a discrete set of instantaneous parameter values of a homogeneous equation over one period. Moreover, it eliminates the need for numerical integration while retaining the benefits of qualitative analyses achievable through algebraic methods.

A simple example involving an analysis of the equations governing a reluctance converter demonstrates the efficiency and unique features of the proposed method.

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