PRZEGLĄD ELEKTROTECHNICZNY

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DOI: 10.15199/48.2025.11.30

Optimization of magnetic field parameters in a system with symmetrical c-shaped poles using conformal mapping

Optymalizacja parametrów pola magnetycznego w układzie z symetrycznymi biegunami typu C z zastosowaniem mapowania konforemnego

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Abstract: The possibility to calculate the intensity of a magnetic field in symmetric C-type poles system based on conformal transformation is shown. The obtained expressions make it possible to solve practical problems of choosing a rational geometry of the poles of two-pole magnetic systems according to the criteria of the maximum of the magnetic field intensity gradient and the product of the magnetic field intensity by its gradient.

Keywords: magnetic field, intensity, conformal transformation, permeance

Streszczenie: W niniejszym artykule pokazano możliwość obliczenia natężenia pola magnetycznego w układzie symetrycznych biegunów typu C. Uzyskane dane umożliwiają rozwiązywanie praktycznych problemów dotyczących wyboru racjonalnej geometrii biegunów w dwubiegunowych układach magnetycznych zgodnie z kryteriami gradientu maksymalnego natężenia pola magnetycznego oraz wyniku natężenia pola magnetycznego.

Słowa kluczowe: pole magnetyczne, natężenie, transformacja konformalna, permeancja

Introduction

Among the variety of electromagnetic systems, a large class is made up of electromagnetic systems, the design of which is based on the shell-type electromagnetic system shown.

A characteristic feature of this system consists in the plane-parallel nature of its magnetic field, which allows the distribution of the magnetic field to be considered as two-dimensional (Fig. 1).

The calculation of such a distribution outside the magnetic circuit in the outer domain of the electromagnet can be carried out by the method of conformal mapping, since in this case it is possible to use the known conformal mapping of the outer domain of a certain rectangle onto the upper half-plane [1, 2]. In our case, such a rectangle is the outer boundary of the electromagnet.

The purpose of this paper is to obtain analytical expressions for calculating the field intensity at any point in the external domain and the total permeance of the magnetic flux from the outer surface of the considered electromagnet, using conformal mapping and specifying boundary conditions that correspond to the adopted plane-parallel design diagram of the magnetic system (Fig. 1). The advantage of analytical expressions for subsequent practical calculations is widely known.

Boundary conditions

We assume, as is usually done when calculating electromagnets, that the outer surface of the magnetic circuit is equipotential. Moreover, in the considered case there are two equipotential surfaces of different polarity (Fig. 2, a): $F_z A_z D_z H_z$ – equipotential surface, which is considered positive; $E_z B_z C_z G_z$ – equipotential surface, which is conditionally considered negative. Here it should be noted that the sign of the surface potential depends on the direction of the current in the magnetization winding. It is also assumed that the lines corresponding to the outer surfaces between

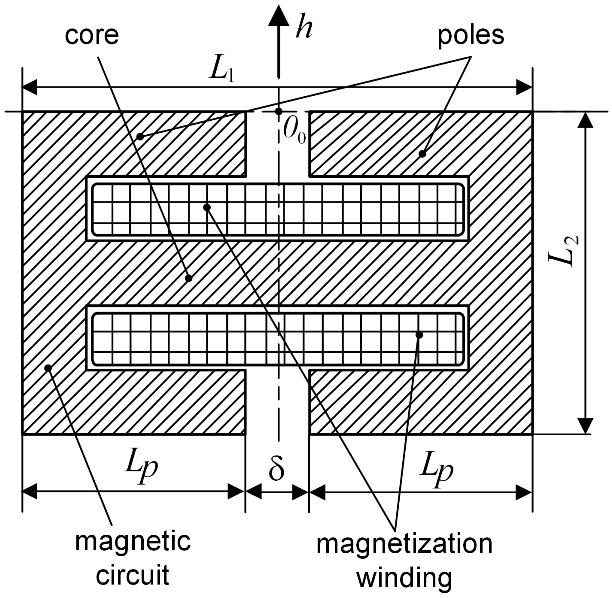

Fig. 1. A designed diagram of the magnetic system of an shell-type electromagnet with external air gaps between the poles

the poles (lines E_zF_z and G_zH_z in Fig. 2, a) are the magnetic lines of force, which is the traditional assumption for such electromagnetic systems.

Thus, there is a system of two symmetrical C-shaped poles, the difference in magnetic potentials between which is taken to be equal to U_0 .

The magnetic field in the outer domain of such a system is calculated by the conformal mapping method. The symmetry of the system is taken into account for this purpose. This makes it possible to first perform a conformal mapping of its outer domain (the domain outside the rectangle $A_zB_zC_zD_z$ in Fig. 2, a) to the upper half-plane with the correspondence of the boundary points shown in Fig. 2, b (the corresponding points of the boundary have the same designation with a subscript corresponding to the designation of the plane, which is given in the circle, Fig. 2).

As a result, we obtain a system of two symmetrical plates of different polarity (Fig. 2, b).

Next, we map the domain above these two plates onto the domain of the uniform field, which can be done by mapping it onto the interior domain of a rectangle whose vertices must correspond to the extreme points of the specified system of two plates.

The function of conformal mapping of the original domain onto the upper half-plane

So, let us find the function of conformal mapping of the outer domain z of rectangle $A_zB_zC_zD_z$ (Fig. 2, a) onto the upper half-plane t (Fig. 2, b).

This function is determined by the well-known Christoffel-Schwarz equation, since the boundary of the mapped domain is a polygon. In this case, the origin of the coordinates of domain z is placed in the center of rectangle $A_zB_zC_zD_z$ (Fig. 2, a) to simplify the form of writing the Christoffel-Schwarz equation.

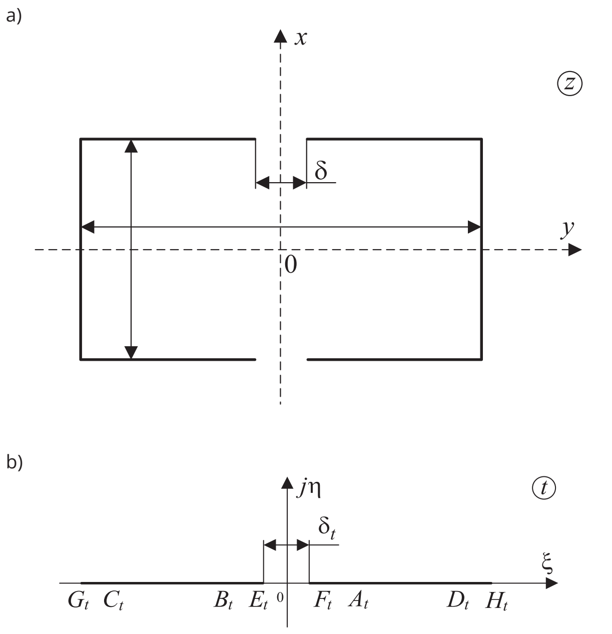

Fig. 2. A design diagram (a) of symmetrical C-shaped poles and correspondence of their boundary points at conformal mapping of the external calculation domain onto the upper half-plane (b)

Then, according to the accepted correspondence of boundary points (Fig. 2), the Christoffel-Schwarz equation can be written in the considered case as

$$\frac{dz}{dt} = S\frac{\sqrt{(t-A_t)(t-B_t)(t-C_t)(t-D_t)}}{\left(t^2+1\right)^2} \tag{1}$$

where S – some mapping constant.

Next, we take into account that in the mapped domain (upper half-plane, Fig. 2, b) the symmetry of the boundary points is preserved: $A_t = -B_t$ and $D_t = -C_t$.

Let us also take into consideration that, according to [1], the following relation holds: $C_t = -1/A_t$.

Then, after replacing variable t by variable

$$\tau = 2 \cdot arctg(-t)\,, \tag{2}$$

from (1) one can obtain by appropriate integration the conformal mapping function $z = f(\tau)$, which, depending on the form of the integration constant, can be written in different ways.

We write down the specified function $z = f(\tau)$ of conformal mapping as

$$z = \frac{1}{4} \cdot \frac{L_1}{E(k) - (k')^2 \cdot K(k)} \times \tag{3}$$

$$\times \left\{ e^{j\tau} - 1 + \sum_{n=1}^{\infty} \frac{a_n}{2n-1} \cdot \left[1 - e^{-j(2n-1)\tau} \right] \right\} + \frac{L_2}{2}.$$

From here on L_1, L_2 – the overall dimensions of the displayed system in terms of the width and height of the poles, respectively (Figs. 1, 2); $K(k)$, $E(k)$ and $K'(k)$, $E'(k)$ – complete elliptic integrals of the first and second kind with moduli, respectively, k and $k' = \sqrt{1-k^2}$, where

$$k = \sin(\alpha), \tag{4}$$

and α – the coordinate of point B_t (Fig. 2,b), expressed through a new variable according to relation (2) as

$$\alpha = 2 \cdot arctg(-B_t); \tag{5}$$

a_n – the coefficients of the series defined by the following recurrence relation

$$(n+1) \cdot a_{n+1} = 2(n-1) \cdot \cos(2\alpha) \cdot a_n - (n-2) \cdot a_{n-1}, \tag{6}$$

for which the terms with zero and one indices are equal, respectively $a_0 = 1$ and $a_1 = -\cos(2\alpha)$.

The ability to determine module k based on known overall dimensions is essential in this case. L_1 and L_2 of the mapped pole system as a result of solving the following transcendental equation [1]

$$\frac{L_1}{L_2} = \frac{E(k) - (k')^2 K(k)}{E'(k) - k^2 K'(k)}, \tag{7}$$

which, taking into account (4), allows determining parameter α as

$$\alpha = \arcsin(k). \tag{8}$$

Determining the size of the plate-pole system

We use the obtained mapping function (3) to determine the coordinates of points E_t, F_t, G_t and H_t according to the known coordinates of the points corresponding to them E_z, F_z, G_z and H_z (Fig. 2). In this case, we take into account that, due to the adopted symmetry of the mapping of the corresponding points of domains z and t, it is enough for the given coordinates of the points F_z and H_z in domain z to find coordinates F_t and H_t in domain t. Then the coordinates of points E_t and G_t, due to the indicated symmetry, are defined as:

$$\left. \begin{aligned} E_t &= -F_t \\ G_t &= -H_t \end{aligned} \right\} \tag{9}$$

As for the coordinates of points F_t and H_t in domain t, using the record of coordinates of the corresponding points in domain z (Fig. 2, a)

$$F_z = \frac{L_2}{2} + j \cdot \frac{\delta}{2} \text{ и } H_z = -\frac{L_2}{2} + j \cdot \frac{\delta}{2},$$

for real numbers of the form

$$\tau_F = 2 \cdot arctg(-F_t) \text{ и } \tau_H = 2 \cdot arctg(-H_t),$$

written according to (2), we can obtain from the conformal mapping equation (3) the following transcendental equation

$$\frac{\delta}{2} = \frac{1}{4} \frac{L_1}{E(k) - (k')^2 k(k)} \times \tag{10}$$

$$\times \left\{ \sin(\tau_{F,H}) + \sum_{n=1}^{\infty} \frac{a_n}{2n-1} \cdot \sin((2n-1)\tau_{F,H}) \right\},$$

which has the same form for τ_F and τ_H, which is reflected in the form of writing this equation.

According to the performed analysis, τ_F and τ_H, as solutions to equation (10), are related to each other by a simple relationship

$$\tau_H = \pi - \tau_F, \tag{11}$$

meaning that, when solving equation (10), it is sufficient to find one of the roots of this equation, for example τ_F, and then use relation (11), which, according to (2), allows determining the coordinates of points F_t and H_t as

$$F_t = -tg(0.5\tau_F) \tag{12}$$

and

$$H_t = -tg(0.5\tau_H). \tag{13}$$

Approximations by polynomials of complete elliptic integrals of the first and second kind given in [3] were used for the numerical implementation of the specified method for determining the dimensions in the considered system of pole plates.

The algorithm of the solution is as follows:

- for the given ratio of overall dimensions L_1 and L_2 module k is determined from the numerical solution of equation (7), and parameter α corresponding to this module is determined according to (8);
- value τ_F or τ_H, which satisfies equation (10) for a given ratio δ / L_1, is determined from the numerical solution of equation (10).

To solve equation (10) numerically, the bisection method can be used, as when ratio δ / L_1 changes in the obvious range from 0 to 1, based on the above, it is easy to indicate the range of values for τ_F and τ_H: the range of value τ_F change – $[0,\alpha]$; the range of value τ_H change – $[\pi, \pi-\alpha]$.

Thus, the above makes it possible to determine the coordinates of the points that define the dimensions of the equipotential plates of a symmetrical two-pole system (Fig. 2, b). The distribution of the magnetic field for it can be easily obtained using also a conformal mapping, as it was done in [1] when determining the field of symmetric charged conducting plates.

Conformal mapping of the field of symmetrical plates onto the uniform field domain

In order to be able to use the corresponding results given in [1], we transform the initial system of two pole plates with a gap between them (Fig. 2, b)

$$\delta_t = 2F_t$$

or, taking into account (9) and (12),

$$\delta_t = 2tg\frac{\tau_F}{2}, \tag{14}$$

into a system of two symmetrical pole plates with a gap between them equal to two.

For this purpose, we use a scaling transformation (it is, as is known, a conformal transformation)

$$T = \frac{2t}{\delta_t} = \frac{t}{tg(0.5\tau_F)}. \tag{15}$$

The resulting correspondence of boundary points is shown in Fig. 3.

According to [1], the upper half-plane T (Fig. 3) can be mapped onto a rectangle in plane W, if we use a transformation of the form

$$T = sn(2K(k')W), \tag{16}$$

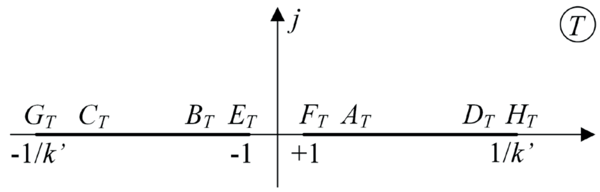

Fig. 3. Scaling transformation

where sn – Jacobi function; W – a complex variable corresponding to the interior of the rectangle onto which the upper half-plane T is mapped; $K(k')$ – elliptic integral of the first kind, whose module k' is given by the relation

$$k' = \delta_t / G_t$$

or, according to (14), (13) and (11),

$$k' = 2\frac{tg(0.5\tau_F)}{tg(0.5(\pi-\tau_F))}. \tag{17}$$

The resulting correspondence of boundary points in domain W is shown in Fig. 4. This figure depicts the last stage of the considered conformal mappings ($z \to t \to T \to W$), as a result of which a domain of a homogeneous field between the vertical pole plates was obtained.

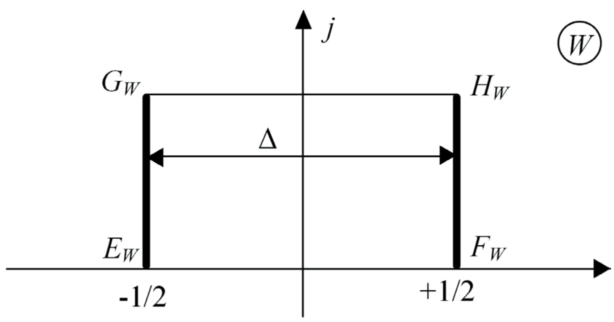

Fig. 4. Domain W of a homogeneous field

Magnetic field intensity in the original system of symmetrical c-shaped poles

According to the theory of conformal mapping, the magnetic field intensity of the mapped domain is related to the field intensity in the mapped domain through the ratio of differentials for the complex coordinates of the corresponding domains [1]. Taking this into account, and also the fact that the field in domain W is homogeneous, its intensity is equal to (Fig. 4)

$$H_W = \frac{U_0}{\Delta} = U_0, \tag{18}$$

where U_0 – the difference in magnetic potentials between equipotentials of different polarities $G_W E_W$ and $H_W F_W$, equal to the difference in magnetic potentials between the poles of the original system; Δ – the distance between these equipotentials ($\Delta = 1$) (Fig.4).

To obtain an analytical expression for intensity H_z in the original domain z we can write the following chain of transformations

$$H_z = H(t) = \cdot \frac{dt}{dz} = H(T) \cdot \frac{dT}{dt} \cdot \frac{dt}{dz} = H_W \cdot \frac{dW}{dT} \cdot \frac{dT}{dt} \cdot \frac{dt}{dz}$$

or, taking into account (18),

$$H_z = U_0 \cdot \frac{dW}{dT} \cdot \frac{dT}{dt} \cdot \frac{dt}{dz}, \tag{19}$$

where $H(t)$ and $H(T)$ – magnetic field intensities in the upper half-planes t and T, respectively.

According to (1), (15) and (16), we can finally write the following expression instead of

$$(19) \quad H_z = \frac{U_0}{D} \cdot \frac{1}{t_F} \cdot \frac{E'(k) - k^2 K'(k)}{k \times K\left(\sqrt{1 - k^2}\right)} \cdot \frac{1}{\sqrt{1 - \left(\frac{t}{t_F}\right)^2}} \times$$

$$(20) \quad \times \frac{1}{\sqrt{1 - k^2 \left(\frac{t}{t_F}\right)^2}} \cdot \frac{\left(t^2 + 1\right)^2}{\sqrt{t^2 - t_A^2} \times \sqrt{t^2 - \left(\frac{1}{t_A}\right)^2}},$$

where $t_A = tg\left(0.5 \arcsin(k)\right)$.

When writing expression (20), we also took into account the properties of the used conformal mappings described above.

Thus, if the dimensions of the initial system of symmetrical C-shaped poles are given, expression (20) allows determining the field intensity H_0 at any point z_0 of the original domain according to the known conformal mapping function (3) and the numerical solution of transcendental equations (7), (9) and the equation obtained from (3) when substituting z_0 into it.

The permeance of the total flux in the outer domain of symmetrical c-shaped poles

Let us take into account that, as is known, the total permeance of magnetic flux in a system of magnetic poles is an analogue of electrical capacitance in a system of identical electrodes. Then, according to [1], the following relationship can be written for the total magnetic flux reduced permeance in the considered case

$$\lambda_\Sigma = \frac{K\left(\sqrt{1 - (k')^2}\right)}{2 \cdot K(k')}, \tag{21}$$

where module k' – of elliptic integral K is defined by expression (17).

Note that the numerical calculation of the field intensity H_z and permeance λ_Σ using the given formulas is not associated with any computational difficulties and can be used to analyze the distribution of the magnetic field in the considered pole system.

The determination of rational geometry of electromagnet poles

As is known, when using a C-shaped electromagnetic system as a traction system in electromagnetic separators, the main interest is in finding the product $H \cdot gradH$ in the working area of the device, which determines its extraction force [4].

Using (3, 19, 20) the following expressions can be written

$$gradH = -\frac{1}{2} \cdot \frac{U_0}{d^3} \cdot \frac{z}{K(k)} \times$$

$$\times \frac{\sqrt{1 + k^2 \cdot \left(\frac{z}{d}\right)^2} + k^2 \sqrt{1 + \left(\frac{z}{d}\right)^2}}{\left[1 + \left(\frac{z}{d}\right)^2\right] \cdot \left[1 + k^2 \cdot \left(\frac{z}{d}\right)^2\right]}, \tag{22}$$

$$H \cdot gradH = -\frac{1}{4} \cdot \frac{U_0^2}{d^4} \cdot \frac{z}{(K(k))^2} \times$$

$$\times \frac{\sqrt{1 + k^2 \cdot \left(\frac{z}{d}\right)^2} + k^2 \sqrt{1 + \left(\frac{z}{d}\right)^2}}{\left[1 + \left(\frac{z}{d}\right)^2\right] \cdot \left[1 + k^2 \cdot \left(\frac{z}{d}\right)^2\right]}, \tag{23}$$

where $K(k)$ – elliptic integral of the first kind of module $k = d / (L_p + d)$, $d = \delta / 2$ (Fig. 1).

Let us introduce into consideration the quantity h, as the distance from the surface of the poles to the desired point on the vertical axis of symmetry of the electromagnetic separator (Fig. 1), which in practical calculations corresponds to the specified depth of extraction of the separator. Then, as a result of direct

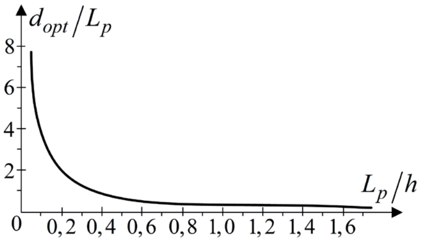

Fig. 5 Dependence $d_{opt} / L_p = f(L_p / h)$ for $gradH$

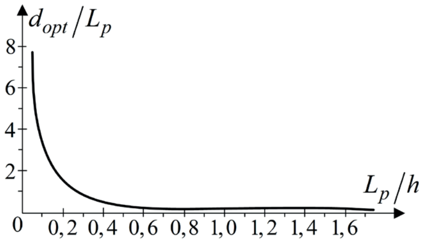

Fig. 6 Dependence $d_{opt} / L_p = f(L_p / h)$ for $H \cdot gradH$

calculations according to (22) and (23), we obtain that the criteria $gradH$ and $H \cdot gradH$, considered as functions d, have a maximum for the given size values h and L_p. This makes it possible to solve the problem of determining such a value d for fixed values h and L_p that satisfies the condition of the maximum of the criteria $gradH$ and $H \cdot gradH$. These problems were solved numerically (as a search for the maximum $gradH$ or $H \cdot gradH$ by simple enumeration of argument d). In this case, the calculated value d, for which the maximum holds $(gradH)_{\max}$ or $(H \cdot gradH)_{\max}$, was accepted as a result of the solution d_{opt}.

The result of solving the specified optimization problems is presented in the form of functional dependences $d_{opt} / L_p = f(L_p / h)$ shown in Figs. 5 and 6.

In practical applications such as electromagnetic separators, the primary objective is to determine the optimal pole geometry that maximizes extraction force. This requires the numerical solution of transcendental equations to determine the magnetic field distribution and its gradient. Similar numerical approaches are widely used in other fields of physics, such as in modeling semiconductor structures [5, 6]. The finite element method is also used to optimize such problems in electromagnetism, enabling the analysis of field distributions in complex geometries [7, 8]. These techniques are indispensable tools in modern engineering, enabling the design of advanced devices without the need for costly physical experiments.

Conclusions

Relations for the numerical calculation of the magnetic field intensity at any point in the outer domain of symmetrical C-pole systems, as well as the total permeance of the magnetic flux of this system have been obtained.

Dependences for choosing a rational ratio of the geometry of the poles of two-pole magnetic systems according to the criterion of the maximum gradient of the magnetic field intensity and the product of the magnetic field intensity by its gradient have been proposed, which is important in practical calculations of separators with C-shaped electromagnetic systems.

Received: 26.10.2025, **Accepted:** 04.11.2025, **Published:** 24.11.2025

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