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A computationally tractable stabilizing predictive controller with fuzzy cost functions for discrete-time nonlinear systems

Łatwy w obliczeniach regulator stabilizujący z funkcją kosztu rozmytego dla dyskretnych układów nieliniowych

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Abstract. A novel stabilizing controller for discrete-time nonlinear systems that uses a fuzzy receding horizon feedback control law is presented. The control input and the state variables are both subject to constraints in this controller. The membership functions of the fuzzy goals and the fuzzy constraints in the fuzzy cost function handles those imposed constraints. We prove asymptotic stability of the infinite horizon fuzzy optimal control law that uses the product as an aggregation operator with respect to time. This result is essential for showing that the finite horizon's closed-loop asymptotic stability can be guaranteed using just the standard final region constraint and, in this case, the switching to the local stabilizing controller will not be necessary inside this final region. The main contribution of the paper is a practical and computationally tractable suboptimal fuzzy receding horizon control scheme that is asymptotically stabilizing. The effectiveness of the proposed scheme is shown by simulations on a continuous stirred tank reactor (CSTR) model.

Keywords: asymptotic stability, model fuzzy predictive control, nonlinear systems, suboptimal control

Streszczenie. Przedstawiono nowatorski regulator stabilizujący dla dyskretnych układów nieliniowych, który wykorzystuje prawo sterowania ze sprzężeniem zwrotnym z rozmytym horyzontem ustępującym. Zarówno sygnał wejściowy sterowania, jak i zmienne stanu podlegają ograniczeniom w tym regulatorze. Funkcje przynależności celów rozmytych i ograniczenia rozmyte w funkcji kosztu rozmytego uwzględniają te narzucone ograniczenia. Dowodzimy asymptotycznej stabilności prawa sterowania optymalnego rozmytego o nieskończonym horyzoncie, które wykorzystuje iloczyn jako operator agregacji względem czasu. Wynik ten jest kluczowy dla wykazania, że asymptotyczna stabilność zamkniętej pętli skończonego horyzontu może być zagwarantowana przy użyciu jedynie standardowego ograniczenia obszaru końcowego, a w tym przypadku przełączanie na lokalny regulator stabilizujący nie będzie konieczne wewnątrz tego obszaru końcowego. Głównym wkładem artykułu jest praktyczny i łatwy w obliczeniach suboptymalny schemat sterowania z rozmytym horyzontem ustępującym, który jest asymptotycznie stabilizujący. Skuteczność proponowanego schematu została wykazana za pomocą symulacji przeprowadzonych na modelu reaktora zbiornikowego z ciągłym mieszaniem.

Słowa kluczowe: stabilność asymptotyczna, rozmyte sterowanie predykcyjne modelu, układy nieliniowe, sterowanie suboptymalne

Introduction

Model predictive control (MPC) is a control technique that has become an accepted standard in the process industries and academia [1, 2, 3, 4]. The success of this control strategy is mainly due to its ability to cope with hard constraints imposed on the state and the control variables. All MPC implementations require the optimization of a cost function. For unconstrained and constrained linear MPC, the use of conventional cost functions, which are based on the 1-norm, the Euclidean 2-norm, and the infinity norm, produces analytical and fast explicit solutions [5, 6, 7]. However, those standard cost functions are not really beneficial when the model is nonlinear because they cause the optimization problem to become generally nonconvex. To translate the specifications of a system in a simpler and more transparent way, Bellman and Zadeh proposed the concept of multistage fuzzy decision making [8]. A thorough overview of multistage fuzzy decision making for systems control was provided in [9]. However, in [9] only open-loop control applications were considered. The use of multistage fuzzy decision making in MPC was introduced in [10] and a method for weighting the importance of goals and constraints in it was addressed in [11]. It is worth noting that in traditional non fuzzy cost MPC, the goals and the constraints are typically defined using crisp, numerical formulations that require precise specifications imposed on the states and also precise constraints imposed on the control inputs. However, in many practical applications, these goals and constraints are not strictly numerically precise; rather, they are often subject to linguistic, vague, or soft requirements. By modelling the objective function using fuzzy goals and fuzzy constraints, the proposed approach introduces a more flexible and expressive framework for defining performance and feasibility criteria. This enables the controller to prioritize trade-offs between competing objectives in a natural and human-like manner, especially under uncertainty or when hard constraints are overly restrictive or impractical. Moreover, the use of fuzzy cost functions in MPC allows the control design to incorporate expert knowledge and linguistic rules directly into the optimization problem, enhancing interpretability and adaptability. This hybridization of fuzzy logic with MPC can bridge the gap between crisp optimal control and soft decision-making, offering a robust framework for systems where tolerances, preferences, or acceptability ranges are not strictly defined.

In systems control, asymptotic stability is the ultimate performance target. For standard non-fuzzy cost nonlinear model predictive control (NMPC), using Lyapunov stability results [12] with set invariance theory [13], closed-loop asymptotic stability is generally ensured by constraining the state of the system to belong to a final invariant set at the end of the prediction horizon and/or adding a final state weight on the finite horizon cost function [14, 15, 16]. Moreover, it was proved that under some mild conditions, suboptimal MPC is asymptotically stabilizing [17]. Hence, research on stability of nonlinear MPC with standard non-fuzzy cost functions have reached a mature stage.

On the other hand, and to the best of the author's knowledge, very few stability research results are available for the MPC with fuzzy cost functions. Indeed, for this class of MPC control, known as model fuzzy predictive control (MFPC), the authors in [18] adopted a dual mode strategy to insure asymptotic stability. This latest was proved under conditions on the final set, the locally stabilizing final control law, which are typical conditions in nonlinear standard cost MPC, and an additional condition on the membership functions of the fuzzy constraints and the fuzzy goals. This latest condition seems to be restrictive and challenging to meet for any invariant final region X_f. In an attempt to relax the latest condition and also avoid the switching to the locally asymptotically stabilizing controller, it has been shown in [19] that the MFPC would be asymptotically stabilizing with a suitable choice of the aggregation operators in the fuzzy cost function. Also, the authors in [20] show that the equilibrium of the infinite horizon MFPC law with the minimum aggregation operator is not asymptotically stabilizing. It is also worth noting that this class of MPC with fuzzy cost functions (MFPC) differs to another class of MPC where the dynamic model under control is fuzzy whereas the cost function is the standard one. This latest class of MPC is known as fuzzy model predictive control (FMPC). Also, for this third class of MPC, research on stability has reached a mature stage (see e.g., [21, 22, 23, 24]).

This paper studies asymptotic stability of MFPC for nonlinear discrete time systems. We will show that the infinite horizon MFPC is asymptotically stabilizing if the fuzzy goals and the fuzzy constraints are aggregated with respect to time using the product operator. Based on this, we will introduce and develop a practical and computationally tractable suboptimal MFPC algorithm that is asymptotically stabilizing. Feasibility of the proposed algorithm at each sampling time will also be studied.

The reminder of the paper is structured as follows. In section II, asymptotic stability of the infinite horizon MFPC law is shown. The suboptimal MFPC algorithm is given in section III; its stabilizing properties and feasibility are then established. Simulation results for an unstable nonlinear system are given in section IV. Finally, section V presents the conclusions.

Asymptotic stability of infinite horizon non-linear MFPC

Consider the discrete-time nonlinear state space model:

$$x(t+1) = f\left[x(t), u(t)\right], \tag{1}$$

where:
$t \geq 0$ – the current time instant,
$x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ – the state and the control input respectively,
$f \in C^2$, and $f(0,0) = 0$, i.e., – the origin is an equilibrium point for the non-linear system.

A fuzzy goal G_i, $i = 1,\ldots,n$, is imposed on each element $x_i(t)$ of the vector $x(t) \in \mathbb{R}^n$, $\forall t \geq 0$ [10]. Similarly, a fuzzy constraint C_j, $j = 1,\ldots m$, is imposed on each

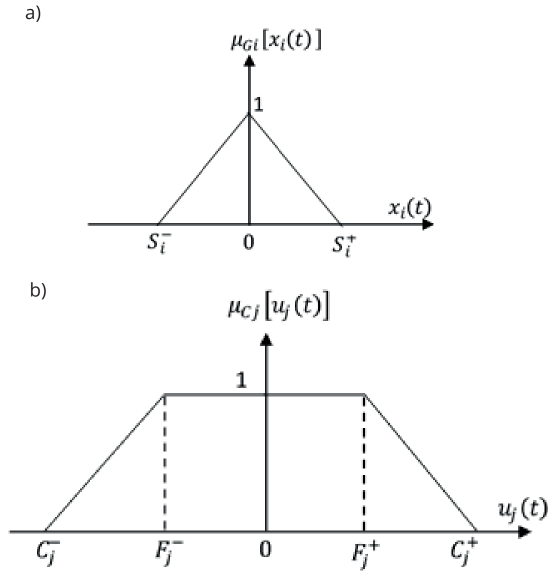

Fig. 1. An example of membership functions: (a) a fuzzy goal and (b) a fuzzy constraint.

element $u_j(t)$ of the vector $u(t) \in \mathbb{R}^m$, $\forall t \geq 0$ [10]. The fuzzy goals and fuzzy constraints are characterized by their membership functions μ_{Gi} and μ_{Cj}. An example of membership functions characterizing fuzzy goals and fuzzy constraints is shown in figure 1.

In the sequel, we suppose that the membership functions of the fuzzy goals and the fuzzy constraints are as shown in figure 1. We also assume that a full measurement of the state vector $x(t)$ is available at each time instant $t \geq 0$. In this case, to apply successfully the MFPC to nonlinear system (1), one has to solve at each sampling time t the following optimization problem:

$$\max_{\{u(t),u(t+1),\ldots,u(t+N-1)\}} J_N\left[x(t),u(t),u(t+1),\ldots,u(t+N-1)\right] =$$

$$\bigwedge_{k=t}^{t+N-1} \left\{\mu_G\left[x(k)\right] * \mu_C\left[u(k)\right]\right\} \tag{2.a}$$

subjct to:

$$x(k+1) = f\left[x(k),u(k)\right], t \leq k \leq t+N-2 \tag{2.b}$$

$$\mu_G\left[x(k)\right] = \mu_{G1}\left[x_1(k)\right] * \mu_{G2}\left[x_2(k)\right] * \ldots * \mu_{Gn}\left[x_n(k)\right]$$
$$, t \leq k \leq t+N-1 \tag{2.c}$$

$$\mu_C\left[u(k)\right] = \mu_{C1}\left[u_1(k)\right] * \mu_{C2}\left[u_2(k)\right] * \ldots * \mu_{Cm}\left[u_m(k)\right]$$
$$, t \leq k \leq t+N-1 \tag{2.d}$$

Note that:

i) N is the prediction horizon.

ii) The function $J_n[x(t), u(t), u(t+1),\ldots,u(t+N-1)]$ given in (2.a) is the fuzzy cost function.

iii) The solution of the optimization problem (2.a) – (2.d), noted $\{u^*(t), u^*(t+1),\ldots, u^*(t+N-1)\}$, is a point where the confluence between the fuzzy goals and the fuzzy constraints is at its maximum.

iv) $\wedge$ and $*$ given in equations (2.a), (2.c) and (2.d) are the aggregation operators which can be the minimum, the product or any suitable triangular norm (t-norm). The operators $\wedge$ and $*$ are usually chosen to be same in MFPC. Here, they can be different since this would allow deriving an asymptotically stabilizing MFPC scheme.

v) At current time t and after solving the optimization problem (2.a) – (2.d), only the first control action $u^*(t)$ of the optimal control sequence $\{u^*(t), u^*(t+1),\ldots, u^*(t+N-1)$ is extracted from the sequence and applied to the plant (1). The other future control actions $u^*(t+1)$, $u^*(t+2)$, …, $u^*(t+N-1)$ are then dropped.

vi) At next time instants, the three steps: – measuring the new current state, – solving the optimization problem (2.a) – (2.d) based on the newly available state, and – extracting and applying only the first control action should all be repeated.

Let us now show that if the aggregation operator $\wedge$, used in the fuzzy cost function (2.a), is substituted with the algebraic product then the infinite horizon MFPC would asymptotically stabilize nonlinear system (1) to the equilibrium. Substituting the prediction horizon N with ∞ and the operator $\wedge$ with the algebraic product into the fuzzy cost (2.a), the optimization problem to be solved at each sampling time t will be given by:

$$\max_{\{u(t),u(t+1),u(t+2),\ldots\}} J_\infty\left[x(t),u(t),u(t+1),u(t+2),\ldots\right] =$$

$$\prod_{k=t}^{\infty} \left\{\mu_G\left[x(k)\right] * \mu_C\left[u(k)\right]\right\} \tag{3.a}$$

subject to:

$$x(k+1) = f\left[x(k),u(k)\right], \quad t \leq k \leq \infty \tag{3.b}$$

$$\mu_G\left[x(k)\right] = \mu_{G1}\left[x_1(k)\right] * \mu_{G2}\left[x_2(k)\right] * \ldots * \mu_{Gn}\left[x_n(k)\right]$$
$$, t \leq k \leq \infty \tag{3.c}$$

$$\mu_C\left[u(k)\right] = \mu_{C1}\left[u_1(k)\right] * \mu_{C2}\left[u_2(k)\right] * \ldots * \mu_{Cm}\left[u_m(k)\right]$$
$$, t \leq k \leq \infty \tag{3.d}$$

Theorem 1: If the state $x(t)$ of system (1) is feasible over an infinite horizon time [19], then the infinite horizon MFPC law asymptotically stabilizes the nonlinear system (1) to the origin.

Proof: Let $K^{IHMFPC}[x(t)]$ be the infinite horizon MFPC law. If $x(t) = 0$, then the optimal sequence $\{u^*(t), u^*(t+1), u^*(t+2),\ldots\}$ solution of (3.a) – (3.d) is equal to zero. Hence, $K^{IHMFPC}(0) = u^*(t) = 0$ and since $f(0,0) = 0$ then $x = 0$ is an equilibrium point for the infinite horizon MFPC closed loop system:

$$x(t+1) = f[x(t), K^{IHMFPC}(x(t))] \tag{4}$$

Now let S_∞ be the set of all feasible states over an infinite horizon time and define the Lyapunov candidate function $V[x(t)] = 1 – J_\infty^*$ [x(t)], where $J_\infty^*[x(t)]$ is the optimal fuzzy cost of the problem (3.a) – (3.d). Since $J_\infty^*(0) = 1$ then $V(0) = 0$. Moreover, for the triangular fuzzy goal shown in figure (1.a), if $x(t) \neq 0$ then $\mu_G[x(t)] < 1$ and hence $J_\infty[x(t), u(t), u(t+1), u(t+2),\ldots] < 1$for any $\{u(t), u(t+1), u(t+2),\ldots\}$, $x(t) \neq 0$. This implies that $V[x(t)] > 0$, $\forall t \in S_\infty$ and $x(t) \neq 0$. Thus $V[x(t)]$ is positive semi-definite on S_∞.

We will prove now that the function $V: S_\infty \to [0,1[$ decreases, along the trajectories of the closed loop system (4), in the following manner: there exists a κ function $\Upsilon: \mathbb{R}^+ \to \mathbb{R}^+$, such that $V[x(t+1)] – V[x(t)] \leq – \Upsilon(\|x(t)\|)$, $\forall x(t) \in S_\infty$.

Let $x(t) \in S_\infty$. In this case, at current time t, the optimal fuzzy cost function is given by:

$$J_\infty^*[x(t)] = J_\infty[x(t), u^*(t), u^*(t+1), u^*(t+2),\ldots] =$$

$$\{\mu_G[x(t)] * \mu_C[u^*(t)]\} \prod_{k=t+1}^{\infty} \{\mu_G[x^*(k)] * \mu_C[u^*(k)]\} \tag{5}$$

where: $\{u^*(t), u^*(t+1), u^*(t+2),\ldots\}$ is the optimal control sequence solution of (3.a) – (3.d), $x^*(k+1) = f[x(k), u^*(k)]$ for $k = t$ and $x^*(k+1) = f[x^*(k), u^*(k)]$ for $k \geq t+1$. Using the principle of optimality [25], [26], the optimal fuzzy cost can be expressed at next time instant as:

$$J_\infty^*[x(t+1)] = \prod_{k=t+1}^{\infty} \{\mu_G[x^*(k)] * \mu_C[u^*(k)]\} \tag{6}$$

From (5), (6) and since $V[x(t)] = 1 – J_\infty^*[x(t)]$, it follows that: $V[x(t+1)] – V[x(t)] = J_\infty^*[x(t)] – J_\infty^*[x(t+1)] = \{\mu_G[x(t)]*\mu_C[u^*(t)] – 1\} J_\infty^*[x(t+1)]$. Also, we have $\mu_G[x(t)] – 1 = 0$ if $x(t) = 0$ and $\mu_G[x(t)] – 1 < 0$ for all other $x(t) \in S_\infty$, $x(t) \neq 0$. Then, there exists a κ function Υ such that $\mu_G[x(t)] – 1 \leq – \Upsilon(\|x(t)\|)$ for all $x(t) \in S_\infty$.

Since:

(i)

$$\mu_G[x(t)]*\mu_C[u^*(t)] – 1 \leq \mu_G[x(t)] – 1 \leq – \Upsilon(\|x(t)\|), \quad \forall x(t) \in S_\infty,$$

and

(ii)

$$0 < \prod_{k=t+1}^{\infty} \{\mu_G[x^*(k)] * \mu_C[u^*(k)]\} \leq 1$$

then $\{\mu_G[x(t)] * \mu_C[u^*(t)] – 1\} J_\infty^*[x(t+1)] \leq -\Upsilon(\|x(t)\|)$. In conclusion, the function $V[x(t)] = 1 – J_\infty^*[x(t)]$ is a Lyapunov function for the closed loop system (4).

The proof of stability presented in theorem 1 is based on the assumption of feasibility. Feasibility of the optimization problem is essential here since if, at a given time instant, the state of the system will be unfeasible this means that the objective function will be equal to zero for any control sequence. This is unacceptable because: (i) the optimization problem will have an infinite number of solutions and (ii) all those solutions will give the worst-case value of the cost function which is zero. That is, there will be at least a fuzzy goal in the fuzzy cost function that is 100% not satisfied whatever is the control sequence. Hence, the feasibility of the state should be considered when deriving fuzzy cost MPC schemes. Moreover, since the infinite horizon MFPC law involves an infinite dimension optimization problem, it is computationally intractable. Nevertheless, it constitutes a touchstone for the finite horizon MFPC case which involves a finite dimension optimization problem at each sampling instant. Indeed, the main difficulty that arises in deriving stabilizing standard non-fuzzy cost MPCs for nonlinear systems is that the optimization problem that must be solved at each instant is only defined over a finite future horizon, yet stability is a property that should hold over an infinite horizon. In this case, the usual way utilized to circumvent this difficulty is to add a terminal constraint that forces the state of the system to belong to a final region X_f at the end of the prediction horizon and add an appropriate terminal cost which depends on this state at the end of the prediction horizon. This terminal cost corresponds to (or approximates) the infinite horizon cost associated with a locally stabilizing controller K_f inside X_f. In this case, the finite horizon MFPC law requires solving, at each time sample, the following optimization problem:

$$\max_{\{u(t),u(t+1),\ldots,u(t+N-1)\}} J_N[x(t), u(t), u(t+1),\ldots,u(t+N-1)]$$

$$= \left\{\prod_{k=t}^{t+N-1} [\mu_G(x(k)) * \mu_C(u(k))]\right\} \mu_{GF}(x(t+N)) \tag{7.a}$$

subject to:

$$x(k+1) = f[x(k), u(k)],$$

$$t \le k \le t + N - 1 \tag{7.b}$$

$$\mu_G[x(k)] = \mu_{G1}[x_1(k)] * \mu_{G2}[x_2(k)] * \ldots * \mu_{Gn}[x_n(k)],$$

$$t \le k \le t + N \tag{7.c}$$

$$\mu_C[u(k)] = \mu_{C1}[u_1(k)] * \mu_{C2}[u_2(k)] * \ldots * \mu_{Cm}[u_m(k)],$$

$$t \le k \le t + N - 1 \tag{7.d}$$

$$x(t+N) \in X_f \tag{7.e}$$

$$\mu_{GF}[u(x(t+N)) =$$

$$\prod_{k=0}^{M-1} \{\mu_G[x(t+N+k)] * \mu_c[K_f(x(t+N+k))]\} \tag{7.f}$$

Note that:

i) The hard constraint (7.e) is the conventional final state constraint imposed on the state at the end of the prediction horizon to ensure asymptotic stability of standard non-fuzzy cost MPCs. In this case, the final region X_f can be calculated here similarly as is usually done in standard MPC (see e.g., [14], [27]).

ii) In equation (7.f), $\mu_{GF}(x(t+N))$ is the final fuzzy goal imposed on the state at the end of the prediction horizon, $M \ge 2$ and $x(t+N+k+1) = f[x(t+N+k), K_f(x(t+N+k))]$, $0 \le k \le M-2$. Hence, at current time instant t, $\mu_{GF}(x(t+N))$ corresponds to the fuzzy cost incurred by the closed loop nonlinear system $x(t+N+k+1) = f[x(t+N+k), K_f(x(t+N+k))]$ from $k = 0$ up to $k = M-2$.

Although the finite horizon MFPC scheme involves a finite dimension optimization problem, it requires calculating the exact solution of the optimization problem (7.a) – (7.f) at each time sample. This optimization problem is non-convex and non-differentiable in general. Therefore, the optimal control sequence $\{u^*(t), u^*(t+1),\ldots,u^*(t+N-1)$ that maximizes, at each sample time t, the fuzzy cost function given in (7.a) subject to (7.b) – (7.f) cannot be easily obtained. To remedy, we will propose in the next section a practical computationally tractable suboptimal model fuzzy predictive control algorithm that is asymptotically stabilizing.

Suboptimal Model Fuzzy Predictive Control

We derive here an implementable Suboptimal Model Fuzzy Predictive Control (SO-MFPC) algorithm which is based on the Branch-and-Bound (B&B) technique [28] and for which assumptions A0-A2 given in [19] are supposed to be satisfied. We also assume that the local stabilizing controller K_f is Lipschitz continuous as is typical in standard NMPC schemes.

SO-MFPC algorithm:

1. Discretize the control space and choose: β such that $0 < \beta \le 1$, and a higher value of the positive integer M.
2. Determine, off-line, a class-κ function θ, and a small ball $B_r \subset X_f$ of radius r around the origin, such that: $\{K_f[x(t)], K_f[x(t+1)],\ldots,K_f[x(t+N-1)]\} \le \theta(\|x(t)\|)$, $\forall x(t) \in B_r$ where $x(t+i+1) = f[x(t+i), K_f[x(t+i)]]$, $i = 0, 1,\ldots, N-2$..
3. At initial sample time instant $t = 0$:
   - 3.1. Find a feasible control sequence $\{u^o(0|0), u^o(1|0),\ldots, u^o(N-1|0)\}$ that generates through (1) a feasible state trajectory $\{x(t+1|t), x(t+2|t),\ldots, x(t+N|t)\}$ such that $(t+N|t) \in X_f$.
   - 3.2. Optimize the previously found control sequence using the B&B technique until the next sample time is reached. The optimized control sequence obtained is denoted here $\{u^o(0|0), u^o(1|0),\ldots, u^o(N-1|0)\}$. It must lead to a feasible state trajectory $\{x^o(1|0), x^o(2|0),\ldots, x^o(N|0)\}$ that satisfies $x^o(N|0) \in X_f$.
   - 3.3. Extract the first control $u^o(0|0)$ and apply it to the plant.
4. At next time instants $t > 0$, and given the previously found optimized control sequence, denoted $\{u^0(t-1|t-1), u^0(t|t-1), u^0(t+N-2|t-1)\}$:

- if the current state $x(t)$ has not reached the small ball B_r yet, then:
  - 4.1. Find a feasible control sequence $\{u(t|t), u(t+1|t),\ldots,(t+N-1|t)\}$ that leads through (1) to a feasible state trajectory $\{x(t+1|t), x(t+2|t),\ldots, x(t+N|t)\}$ that satisfies $x(t+N|t) \in X_f$, and a cost increase given by:

$$J_N[x(t), u(t|t), u(t+1|t),\ldots, u(t+N-1|t)] \ge$$

$$\ge J_N[x(t-1), u^0(t-1|t-1),\ldots, u^0(t+N-2|t-1)] +$$

$$+ \beta\,\{1 - \mu G[x(t-1)] * \mu C[u^0(t-1|t-1)]\}\, P(t|t-1) \tag{8}$$

where:

$$P(t\mid t-1)=\prod_{k=t}^{t+N-2}\left\{\mu_G\left[x^o(k\mid t-1)\right]*\right.$$

$$\left.\mu_C\left[u^o(k\mid t-1)\right]\right\}\prod_{k=t+N-1}^{t+N+M-2}\left\{\mu_G\left[x^o(k\mid t-1)\right]*\right.$$

$$\left.\mu_C\left[K_f\left(x^o(k\mid t-1)\right)\right]\right\} \tag{9}$$

This control sequence is found using the B&B strategy and the sequence

$$\left\{\begin{matrix}u^o(t\mid t-1),u^o(t+1\mid t-1),\ldots,\\ u^o(t+N-2\mid t-1),K_f\left[x^o(t+N-1\mid t-1)\right]\end{matrix}\right\}$$

as an initial guess.

4.2. Optimize this sequence subject to the state constraint $x(t+N\mid t)\in X_f$ until the next sampling time instant is reached. The resulting optimized control sequence would be noted $\{u^0(t\mid t), u^0(t+1\mid t),\ldots, u^0(t+N-1\mid t)\}$

4.3. Extract the first control $u^0(t\mid t)$ and apply it to the plant.

- else if the current state $x(t)$ has reached the small ball B_r, then:
  - Repeat the previous step 4.1. under the additional constraint $\|u(t), u(t+1),\ldots,u(t+N-1)\|\le\theta(\|x(t)\|)$ and the initial guess $\{K_f\{x(t)], K_f[x(t+1)],\ldots,K_f[x(t+N-1)]\}$.
  - Repeat the previous step 4.2. under the additional constraint $\|u(t), u(t+1),\ldots,u(t+N-1)\|\le\theta(\|x(t)\|)$.
  - Repeat the previous step 4.3.
  - end if.

end of the SO-MFPC algorithm.

It is important to note that:

i) In the step 2 of the algorithm, the existence of a class-κ function θ, and a small ball $B_r\subset X_f$, such that $\{K_f[x(t)], K_f[x(t+1)],\ldots,K_f[x(t+N-1)]\}\le\theta(\|x(t)\|)$, $\forall x(t)\in B_r$ is guaranteed given that: (a) the locally stabilizing controller K_f is Lipschitz continuous, (b) the closed-loop system is asymptotically stable under K_f within X_f (see assumptions A0-A2 given in [19]), and (c) the ball B_r is contained in the terminal region X_f.

ii) The aim of three last commands of the algorithm situated after the else if condition is to ensure that $\|u(t), u(t+1),\ldots,u(t+N-1)\|\le\theta(\|x(t)\|)$, $\forall x(t)\in B_r$. This condition is required to prove the asymptotic stability of the SO-MFPC algorithm.

iii) Once the state $x(t)$ reaches the small ball B_r, then the initial guess choice: $\{K_f[x(t)], K_f[x(t+1)],\ldots K_f[x(t+N-1)]$ is (a) a feasible control sequence, (b) generates a feasible state trajectory, (c) keeps the system inside the positively invariant set X_f, (d) satisfies the sub-optimality asymptotic stability condition: there exists a ball B_r and a κ function $\theta(\cdot)$ such that $\|u(t), u(t+1),\ldots,u(t+N-1)\|\le\theta(\|x(t)\|)$, $\forall x(t)\in B_r$. In addition, K_f is asymptotically stabilizing inside X_f. Hence, asymptotic stability and recursive feasibility can be guaranteed using, at each sampling time, only this initial guess sequence once the state $x(t)$ reaches the small ball B_r. In our case, the algorithm continues looking for a better optimal solution that satisfies all required constraints using B&B technique until next sampling time is reached.

We establish in the following theorem the asymptotically stabilizing property of the SO-MFPC algorithm law denoted hereafter $K^{SOMFPC}(x(t))]$.

Theorem 2: Let D be the set of all initial states $x(0)$ for which the SO-MFPC algorithm is feasible over N at each sampling time $t\ge 0$ [19]. Then, the SO-MFPC law asymptotically stabilizes the system (1) in D.

Proof:

Let us define the function:

$$V[x(t), u^0(t\mid t), u^0(t+1\mid t),\ldots, u^0(t+N-1\mid t)] =$$

$$= 1-J_N[x(t), u^0(t\mid t), u^0(t+1\mid t),\ldots, u^0(t+N-1\mid t) \tag{10}$$

where J_N is the fuzzy cost function given in (7.a) and (7.f).

Since:

i) The fuzzy cost J_N is continuous at the origin. So the function $V[x(t), u^0(t\mid t),\ldots,u^0(t+N-1\mid t)]$ given in (10) will also be continuous at the origin. In addition, $V(0)=0$ and $V[x(t), u^0(t[\, t),\ldots, u^0(t+N-1\mid t)] > 0$ for all $[x(t), u^0(t\mid t),\ldots, u^0(t+N-1\mid t)]\neq 0$. In this case, there exists a κ function $\lambda(\cdot)$ such that $V[x(t), u^0(t\mid t),\ldots, u^0(t+N-1\mid t)]\ge\lambda(\|x(t), u^0(t\mid t),\ldots, u^0(t+N-1\mid t)\|)\ge\lambda(\|x(t)\|]$

ii) At time instant t, the algorithm makes:

$$V[x(t), u^0(t+1\mid t),\ldots, u^0(t+N-1\mid t) =$$

$$= 1-J_N[x(t), u^0(t\mid t), u^0(t+1\mid t), \ldots, u^0(t+N-1\mid t)] +$$

$$+\beta\{1-\mu_G[x(t-1)]*\mu_C[u^0(t-1|t-1)]\}\,P(t|t-1). \qquad (11)$$

Also, since: (a) by assumption $\beta > 0$, (b) $P(t|t-1) > 0$ if both the control sequence and the resulting state trajectory are feasible, (c) $1-\mu_G(0)*\mu_C(0) = 0$, and (d) $1-\mu_G\{x(t-1)*\mu_C[u^0(t-1|t-1)] > 0$, for all $x(t-1) \neq 0$ and $u^0(t-1|t-1) \neq 0$; then there exists a κ function $\varphi(\cdot)$ such that:

$$\beta\left\{1-\mu_G\left[x(t-1)\right]*\mu_C\left[u^o(t-1|t-1)\right]\right\}P(t|t-1) \geq \varphi\left(\left\|x(t-1),u^o(t-1|t-1)\right\|\right) \qquad (12)$$

In view of (11) and (12) we conclude that there exists a κ function $\varphi(\cdot)$ such that:

$$V\left[x(t),u^o(t|t),\ldots,u^o(t+N-1|t)\right]-V\left[x(t-1),u^o(t-1|t-1),\ldots,u^o(t+N-2|t-1)\right] \leq -\varphi\left(\left\|x(t-1),u^o(t-1|t-1)\right\|\right)$$

iii) The SO-MFPC algorithm ensures the following condition: there exists a ball B_r and a κ function $\theta(\cdot)$ such that $\|u(t), u(t+1),\ldots,u(t+N-1)\| \leq \theta(\|x(t)\|)$, $\forall x(t) \in B_r$.

Then, in view of (i), (ii) and (iii), the three conditions required for the asymptotic stability of the suboptimal nonlinear MPC law are satisfied [17].

The existence, at each sampling time $t > 0$, of a feasible control sequence that leads to a feasible state trajectory satisfying the final state constraint $x(t+N) \in X_f$ and the cost increase (8) is established in the following theorem.

Theorem 3: Assume that (i) the system given in (1) is controlled using the SO-MFPC algorithm, (ii) the initial state $x(0)$ is feasible over N, and (iii) the predictive control sequence satisfies $\|u(t), u(t+1),\ldots, u(t+N-1)\| \leq \theta(\|x(t)\|)$, $\forall x(t) \in B_r$. Then, the set X_N of all feasible initial states is positively invariant for the closed-loop system $x(t+1) = f[x(t), K^{SOMFPC}(x(t))]$.

Proof: Suppose that $x(t) \in X_N$, and let $\{x^0(t|t), x^0(t+1|t),\ldots,x^0(t+N-1|t)\}$ be the suboptimal control sequence obtained, at time sample t, using the SO-MFPC algorithm and let $\{x^0(t+1|t), x^0(t+2|t),\ldots, x^0(t+N|t)\}$ be the corresponding state trajectory. At next time sample $t+1$, assuming that assumptions A0–A2 given in [19] hold and since the control sequence is constrained to steer $x^0(t+N|t)$ into X_f, then the initial guess noted hereafter $\hat{u}(t+1|t+1), \hat{u}(t+2|t+1),\ldots, \hat{u}(t+N|t+1)$ which is equal to $\{u^0(t+1|t), u^0(t+2|t),\ldots, u^0(t+N-1|t), K_f[x^0(t+N|t)$: (i) is feasible, and (ii) generates through (1) a feasible state trajectory also noted $\hat{x}(t+2|t+1), \hat{x}(t+3|t+1),\ldots, \hat{x}(t+N|t+1)$, $\hat{x}(t+N+1|t+1)$ that also satisfy the final state constraint $\hat{x}(t+N+1|t+1) \in X_f$.

We also have:

$$\begin{aligned}&J_N\left[\hat{x}(t+1|t+1),\hat{u}(t+1|t+1),\ldots,\hat{u}(t+N|t+1)\right]-\\&J_N\left[x^o(t|t),u^o(t|t),\ldots,u^o(t+N-1|t)\right]=\\&\left\{1-\mu_G\left[x^o(t|t)\right]*\mu_C\left[u^o(t|t)\right]\right\}\\&\prod_{k=t+1}^{t+N-1}\left\{\mu_G\left[x^o(k|t)\right]*\mu_C\left[u^o(k|t)\right]\right\}\mu_{GF}\left[x^o(t+N|t)\right]\geq\\&\beta\left\{1-\mu_G\left[x^o(t|t)\right]*\mu_C\left[u^o(t|t)\right]\right\}\\&\prod_{k=t+1}^{t+N-1}\left\{\mu_G\left[x^o(k|t)\right]*\mu_C\left[u^o(k|t)\right]\right\}\mu_{GF}\left[x^o(t+N|t)\right]\end{aligned}$$

Hence the initial guess

$$\left\{\hat{u}(t+1|t+1),\hat{u}(t+2|t+1),\ldots,\hat{u}(t+N|t+1)\right\}$$

satisfies also the cost increase (8).

Note that if $M \to \infty$, then feasibility of the initial state $x(0)$ over N guarantees the asymptotic stability of the SO-MFPC algorithm.

Illustrative Example

As an example, we consider the CSTR system [19]. The system is described by the following dynamic model equations:

$$\dot{x}_1 = x_2 + 0.5(1+x_1)u \qquad (13.a)$$

$$\dot{x}_2 = x_1 + 0.5(1+x_2)u \qquad (13.b)$$

Note that the origin is an unstable equilibrium point.

The aim here is to stabilize the state $x = [x_1 \;\; x_2]^T$ of the system to the origin as fast as possible and in addition, the control input should satisfy the constraint $-2 \leq u \leq 2$. To do that, two control techniques have been applied to the CSTR system (13): the proposed Suboptimal Model Fuzzy Predictive Control Algorithm and the standard asymptotically stabilizing Nonlinear Model Predictive Controller with the conventional quadratic cost. The prediction horizon is $N = 11$

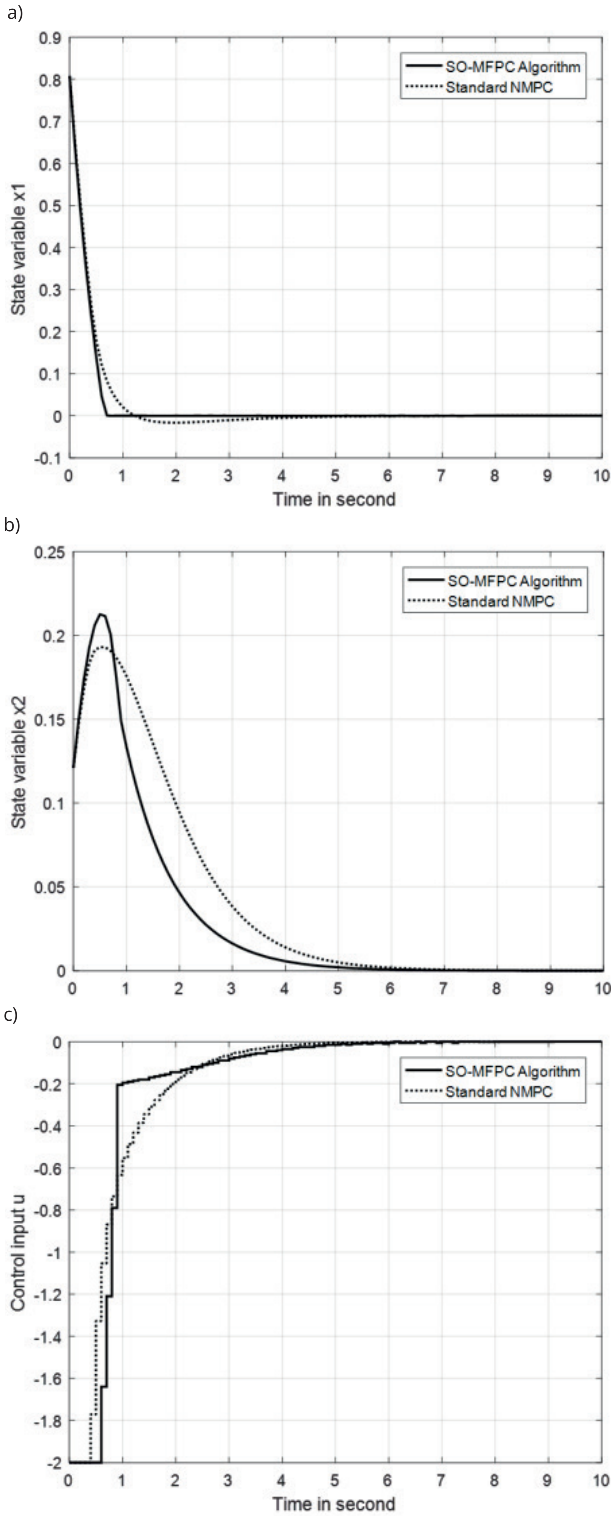

Fig. 2. Closed-loop control simulation results using the SO-MFPC algorithm and the standard NMPC technique: (a) state $x_1(t)$, (b) sate $x_2(t)$, and (c) control input $u(t)$.

for both compared predictive controllers, and the sampling time is $T_s = 0.1$. Also, the terminal invariant set is $X_f = \{x \in \mathbb{R}^2 / x^T Px \le 0.7\}$ with

$P = \begin{bmatrix} 16.5926 & 11.5926 \\ 11.5926 & 16.5926 \end{bmatrix}$, the local stabilizing control law $K_f(x) = [-2.118 \quad -2.118]x$, and the initial condition is $x(0) = [0.808 \quad 0.121]^T$ (see [29]).

For the proposed SO-MFPC algorithm, we take $\beta = 0.01$, $M = 10^4$, and the aggregation operator $*$ to be the parametrized Yager t-norm. Also, to implement the B&B technique, the control space was discretized at each sampling time with a step $\Delta u = 0.02$. The membership functions of the fuzzy goals and the fuzzy constraint have been chosen as shown in figures 1.a and 1.b, respectively. Since the control input is constrained to satisfy the condition $-2 \le u \le 2$, the limits of the fuzzy constraint are taken $c^- = -2$ and $c^+ = 2$.

The other parameters, namely: the Yager's t-norm parameter, S_1^-, S_1^+, S_2^-, S_2^+, F^-, and F^+ for the SO-MFPC and the matrices Q and R for the standard quadratic cost NMPC are obtained using a hyperparameter Bayesian optimization method such that the response of the system should be as fast as possible. Figure 2 shows simulation results for both compared control methods. As can be seen, the control input constraint is satisfied for both control methods. However, the response of the system is clearly converging faster to the equilibrium for the proposed control algorithm.

Conclusion

In this paper, asymptotic stability results for MFPC were given. We have shown that the infinite horizon MFPC is asymptotically stabilizing if the aggregation operator with respect to time is the algebraic product. This result can be used to derive sufficient conditions that guarantee closed-loop asymptotic stability for the non-switching finite horizon MFPC case. We provided a practical computationally tractable feasible suboptimal MFPC algorithm based on B&B that is asymptotically stabilizing. The efficiency of the proposed algorithm has been demonstrated through simulation on a CSTR dynamic system model. Other interesting research topics are the investigation of asymptotic stability conditions with respect to other aggregation operators and membership functions and also study the robustness of the MFPC law.

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